Unit F3
Integration

Introduction

In this unit you will study the question:

> What do we mean by the area between the graph of a real function f and the x-axis?

You will see how this can be answered by trapping the required area between increasingly accurate lower and upper estimates, each of which is the sum of the areas of suitably chosen rectangles. The area between the graph $y = f(x)$ and the segment $[a, b]$ of the x-axis is *defined* to be the supremum of the lower estimates and the infimum of the upper estimates, as long as these two values are equal. In this case, we call the common value the *integral* of f on $[a, b]$, written

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

You will see that, for many functions, we can evaluate integrals by using the *Fundamental Theorem of Calculus*, which allows us to think of integration as the inverse operation of differentiation. Although we will review techniques of integration, our main focus in this unit is on providing a rigorous foundation for the idea of integration, and on showing how this relates to concepts you have met in previous analysis units.

Often it is not possible to evaluate an integral explicitly, and later in the unit you will meet methods for obtaining upper and lower bounds for the integral in such cases. You will also see how we can apply integration to derive some remarkable formulas for π and for estimating factorials, and to give a useful additional test for the convergence of certain series.

1 The Riemann integral

The purpose of this section is to give a rigorous definition of what we mean by the area between the graph

$$y = f(x) \quad (x \in [a, b])$$

and the closed interval $[a, b]$ on the x-axis, and to explore its implications. We begin with an informal discussion to set the scene.

We have an intuitive notion of area. There are formulas for calculating the areas of simple geometric shapes such as rectangles and triangles, and we would certainly want our rigorous definition to agree with these. We would probably also agree that the region between the x-axis and the graph of a *continuous* function defined on a closed interval $[a, b]$ has a definite area, even if we are uncertain how to measure it. On the other hand, for discontinuous functions it is not obvious that we can always say that the region between a graph and the x-axis *has* an area. For example, can we define such an area for the function

$$f(x) = \begin{cases} x^2, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2, \end{cases}$$

which has a discontinuity at the point $x = 1$? The graph of f is illustrated in Figure 1(a).

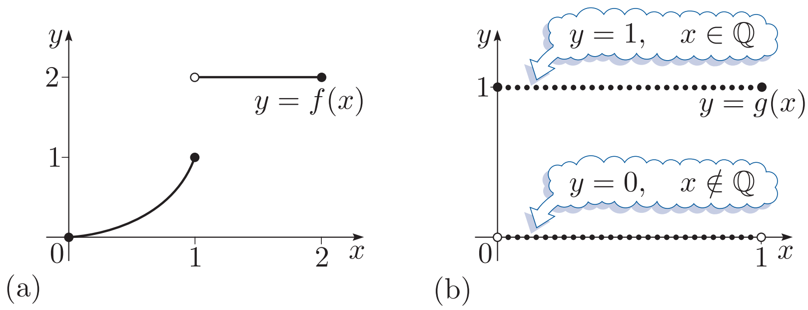


Figure 1 The graphs of the discontinuous functions f and g

As another example, can we define the area between the x-axis and the graph of the **Dirichlet function** on the closed interval $[0, 1]$, illustrated in Figure 1(b)? You met the Dirichlet function in Subsection 3.2 of Unit F1 *Limits*. Its rule is

$$g(x) = \begin{cases} 1, & 0 \leq x \leq 1, \ x \text{ rational}, \\ 0, & 0 \leq x \leq 1, \ x \text{ irrational}, \end{cases}$$

and in Unit F1 it was shown to be discontinuous at every point of its domain.

It seems desirable that our definition of area should cover a wide range of functions. Later in this section we will prove that we can always assign a value to the area between the graph and the x-axis for a continuous function defined on a closed interval. You will also see that we can do the same for the function illustrated in Figure 1(a), but *not* for the function illustrated in Figure 1(b).

Our definition of area is based on finding lower and upper estimates for the 'area' (if it exists) of the region between the graph $y = f(x)$ and the x-axis, using the following approach. First we divide the interval $[a, b]$ into a set of subintervals, called a *partition* of $[a, b]$. Then we consider two sets of rectangles, each rectangle having one of the subintervals as its base. In one set, we choose rectangles whose top edges lie on or below the graph, so

the sum of their individual areas forms a lower estimate for the 'area' of the region; see Figure 2(a). In the other, we choose rectangles whose top edges lie on or above the graph, so the sum of their individual areas forms an upper estimate for the 'area'; see Figure 2(b).

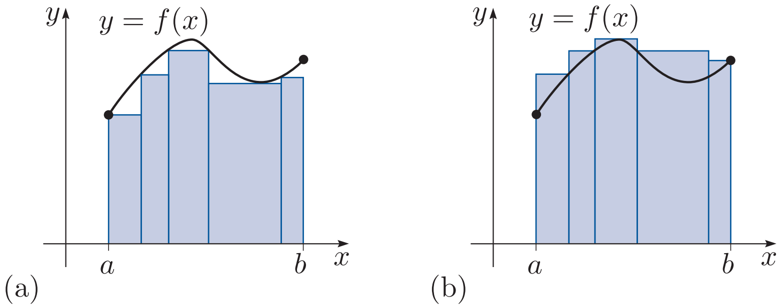


Figure 2 (a) A lower estimate and (b) an upper estimate for the area of the region between the graph $y = f(x)$ and the x-axis

In this way we can obtain many lower estimates and upper estimates by choosing different partitions of $[a, b]$. If there is a real number A with the properties

the supremum of the lower estimates $= A$

and

the infimum of the upper estimates $= A$,

then we define A to be the *area* between the graph and the x-axis. We call the number A the *integral* of f on $[a, b]$, and denote it by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x)\,dx.$$

We make all these ideas precise in the rest of this section, which is the longest and hardest section of the unit. On a first reading, you may wish to try to understand the main ideas without following every detail. The details may be easier to understand on a second reading.

1.1 Definition of the integral

In this subsection we work towards giving a rigorous definition of the area between the graph of a function f defined on a closed interval $[a, b]$ and the x-axis; that is, the integral of f on $[a, b]$.

Before we can give the definition, we need to introduce a number of key ideas. In the paragraphs below you will study:

- some important terminology for functions
- what is meant by a partition of a closed interval
- the use of lower and upper Riemann sums to estimate areas.

Terminology for functions

In Unit D1 *Numbers* you met the definitions of lower bound, greatest lower bound, upper bound and least upper bound of *sets* in $\mathbb{R}$. Here you will meet analogous definitions for *functions* defined on an interval in $\mathbb{R}$. These definitions of greatest lower bound and least upper bound generalise the notion of the *minimum* and *maximum* of a function which you met in Unit D4 *Continuity*. The minimum and maximum of a function are illustrated in Figure 3 and we give a reminder of their definitions (in a slightly different form from those you saw in Unit D4) together with the definitions of lower and upper bounds for functions.

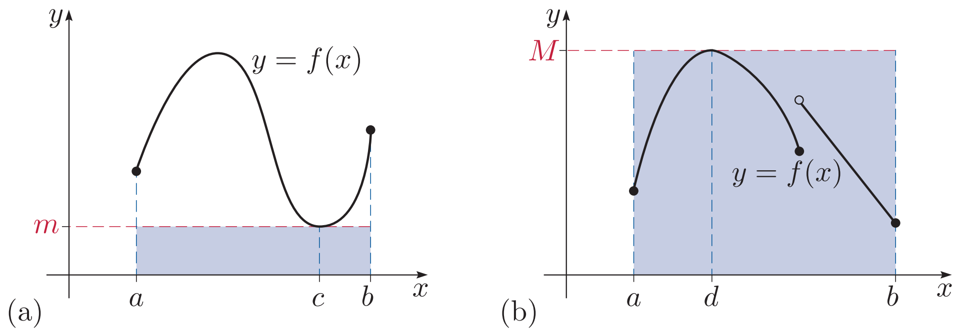


Figure 3 (a) The minimum of a function (b) The maximum of a different function

Definitions

Let the function f be defined on the closed interval $[a, b]$. Then the following hold on $[a, b]$.

- f is **bounded below** on $[a, b]$ with m as a **lower bound** if

  $$f(x) \geq m, \quad \text{for all } x \in [a, b].$$

- m is the **minimum** of f on $[a, b]$ if
  1. m is a lower bound for f on $[a, b]$, and
  2. $f(c) = m$, for some $c \in [a, b]$.

  Thus $m = \min\{f(x) : a \leq x \leq b\}$, which we also write as $\min\limits_{[a,b]} f$ or simply as $\min f$.

- f is **bounded above** on $[a, b]$ with M as an **upper bound** if

  $$f(x) \leq M, \quad \text{for all } x \in [a, b].$$

- M is the **maximum** of f on $[a, b]$ if
  1. M is an upper bound for f on $[a, b]$, and
  2. $f(d) = M$, for some $d \in [a, b]$.

  Thus $M = \max\{f(x) : a \leq x \leq b\}$, which we also write as $\max\limits_{[a,b]} f$ or simply as $\max f$.

- f is **bounded** on $[a, b]$ if it is both bounded below and bounded above on $[a, b]$.

Note that any lower bound for a function f on $[a, b]$ is less than or equal to any upper bound for f on $[a, b]$.

A function f that is continuous on a closed interval $[a, b]$ necessarily has both a minimum and a maximum, by the Extreme Value Theorem for continuous functions, as you saw in Subsection 3.3 of Unit D4. However, if f is not continuous on $[a, b]$, then it may or may not have a minimum or a maximum on $[a, b]$; for example, the function in Figure 4(a) has neither a minimum nor a maximum on $[a, b]$, whilst the function in Figure 4(b) has a maximum but no minimum on $[a, b]$.

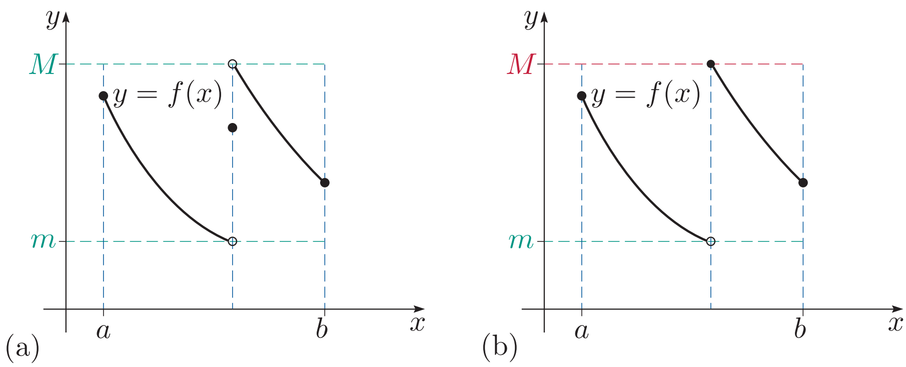


Figure 4 Two bounded functions on $[a, b]$: (a) a function with neither a minimum nor a maximum (b) a function with a maximum but no minimum

Both functions in Figure 4 are certainly bounded on $[a, b]$; and there are numbers m and M, as shown, such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. However, although the first function f takes values as close as we please to m and M, there is no point x in $[a, b]$ where $f(x) = m$ or M; and although the second function f takes values as close as we please to m, there is no point x in $[a, b]$ where $f(x) = m$. This suggests the notions of greatest lower bound and least upper bound.

Definitions

Let the function f be defined on the closed interval $[a, b]$. Then

- m is the **infimum** or **greatest lower bound** of f on $[a, b]$ if
  1. m is a lower bound for f on $[a, b]$, and
  2. if $m' > m$, then $f(c) < m'$, for some $c \in [a, b]$.

  Thus $m = \inf\{f(x) : a \leq x \leq b\}$, which we also write as $\inf\limits_{[a,b]} f$ or simply as $\inf f$.
- M is the **supremum** or **least upper bound** of f on $[a, b]$ if
  1. M is an upper bound for f on $[a, b]$, and
  2. if $M' < M$, then $f(d) > M'$, for some $d \in [a, b]$.

  Thus $M = \sup\{f(x) : a \leq x \leq b\}$, which we also write as $\sup\limits_{[a,b]} f$ or simply as $\sup f$.

Remarks

1. Note from the definition that the concepts of the infimum of a real *function* and the infimum of a *set* of real numbers are related in the following way: the infimum of a function is the infimum of the image set of the function. A similar remark applies to supremums.
2. Any function f that is bounded on $[a, b]$ necessarily possesses an infimum and a supremum on $[a, b]$: if m is any lower bound of f on $[a, b]$, then $\inf f \geq m$ and if M is any upper bound of f on $[a, b]$, then $\sup f \leq M$.
3. If $\min\limits_{[a,b]} f$ exists, then $\inf\limits_{[a,b]} f = \min\limits_{[a,b]} f$. Similarly, if $\max\limits_{[a,b]} f$ exists, then $\sup\limits_{[a,b]} f = \max\limits_{[a,b]} f$.

Worked Exercise F24

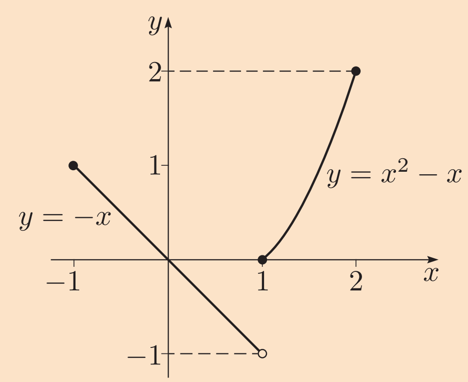
Consider the function

$$f(x) = \begin{cases} -x, & -1 \leq x < 1, \\ x^2 - x, & 1 \leq x \leq 2. \end{cases}$$

Sketch the graph of f and identify $\min f$, $\max f$, $\inf f$ and $\sup f$ (if they exist).

Solution

The graph of f is shown below.



The graph suggests that $\inf f$ is equal to -1 but that $\min f$ does not exist because f does not take the value -1. We begin by checking whether this is true.

First, $\inf f = -1$, since

1. $f(x) \geq -1$, for all $x \in [-1, 2]$,
2. if $m' > -1$, then m' is not a lower bound for f on $[-1, 2]$ because the sequence $(1 - 1/n)$ is contained in $[-1, 2]$ and

$$f(1 - 1/n) = -1 + 1/n \to -1 \quad \text{as } n \to \infty,$$

so there exists $x' \in [-1, 2]$ such that $f(x') < m'$.

Recall that, if $\min f$ exists, then it is equal to $\inf f$.

Also, $\min f$ does not exist since there is no point x such that $f(x) = -1$.

Next, $\max f = 2$, since

1. $f(x) \leq 2$, for all $x \in [-1, 2]$,
2. $f(2) = 2$.

Finally, $\sup f = 2$, since f has maximum 2 on $[-1, 2]$.

Exercise F33

For each of the following functions f on $[-1, 1]$, sketch the graph of f and identify $\min f$, $\max f$, $\inf f$ and $\sup f$ (if they exist).

(a) $f(x) = \begin{cases} x^2, & -1 < x < 1, \\ \frac{1}{2}, & x = \pm 1. \end{cases}$

(b) $f(x) = \begin{cases} x^2, & -1 \leq x \leq 0, \\ x^2 - 1, & 0 < x \leq 1. \end{cases}$

Partitions of a closed interval

We now introduce the notion of a *partition* of a closed interval, which will play an important role in estimating the area between the graph of a function and the x-axis.

Definitions

A **partition** P of a closed interval $[a,b]$ is a collection of a finite number of closed subintervals of $[a,b]$,

$$P = \{[x_0,x_1],[x_1,x_2],\ldots,[x_{i-1},x_i],\ldots,[x_{n-1},x_n]\},$$

where

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_n = b.$$

The points x_i, $0 \le i \le n$, are called the **partition points** of P.

The ith **subinterval** is $[x_{i-1},x_i]$, $1 \le i \le n$, and its **length** is denoted by $\delta x_i = x_i - x_{i-1}$.

The **mesh** of P is the quantity $\|P\| = \max\limits_{1\le i\le n}\{\delta x_i\}$.

A **standard partition** is a partition with subintervals of equal length.

These definitions are illustrated in Figure 5.

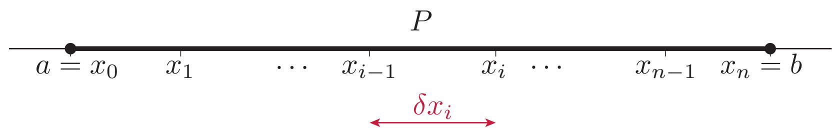


Figure 5 A partition P of an interval $[a,b]$

Worked Exercise F25

Let P be the partition of [0,1] given by

$$P = \left\{\left[0,\tfrac{1}{2}\right],\left[\tfrac{1}{2},\tfrac{3}{5}\right],\left[\tfrac{3}{5},\tfrac{3}{4}\right],\left[\tfrac{3}{4},1\right]\right\}.$$

Find the mesh of P.

Solution

The partition points are 0, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{3}{4}$ and 1.

We have

$$\delta x_1 = \tfrac{1}{2} - 0 = \tfrac{1}{2},\ \delta x_2 = \tfrac{3}{5} - \tfrac{1}{2} = \tfrac{1}{10},\ \delta x_3 = \tfrac{3}{4} - \tfrac{3}{5} = \tfrac{3}{20},$$
$$\delta x_4 = 1 - \tfrac{3}{4} = \tfrac{1}{4}.$$

The mesh of P is the length of the largest subinterval.

So the mesh of P is

$$\|P\| = \max\left\{\tfrac{1}{2},\tfrac{1}{10},\tfrac{3}{20},\tfrac{1}{4}\right\} = \tfrac{1}{2}.$$

P is not a standard partition of $[0,1]$, since not all its subintervals are of equal length.

Exercise F34

Write down the standard partition P of $[-1, 2]$ that contains four subintervals, and state the mesh of P.

Lower and upper Riemann sums

Next, we introduce the *lower* and *upper Riemann sums* for a bounded function f on an interval $[a, b]$ with partition P; these correspond to underestimates and overestimates in our intuitive notion of the area between the graph $y = f(x)$ and the x-axis.

Definitions

Let f be a bounded function on $[a, b]$, and let P be the partition $\{[x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]\}$, where $x_0 = a$ and $x_n = b$. Let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$$

and

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\},$$

for $i = 1, 2, \ldots, n$.

Then the **lower Riemann sum** for f on $[a, b]$ with partition P is

$$L(f, P) = \sum_{i=1}^{n} m_i \delta x_i,$$

and the **upper Riemann sum** for f on $[a, b]$ with partition P is

$$U(f, P) = \sum_{i=1}^{n} M_i \delta x_i.$$

Note that the above definitions work equally well whether f takes positive or negative values in $[a, b]$. Regions between the graph $y = f(x)$ and the x-axis where f takes negative values make a negative contribution to the lower and upper Riemann sums. In this subsection we will in general illustrate results for functions that are *non-negative* throughout $[a, b]$, but we give further consideration to functions that are negative on all or part of the interval $[a, b]$ in Subsection 1.3.

The terms m_i and M_i in the definitions denote the greatest lower bound and least upper bound of f on the ith subinterval of the partition; we need to use the infimum and supremum on the subintervals since the function f may not be continuous and so may not have a minimum or maximum on all (or any) subintervals. The lower Riemann sum is the sum of the areas of the rectangles with height m_i and width δx_i, giving a *lower* estimate for the area between the graph and the x-axis, as illustrated in Figure 6(a). The upper Riemann sum is the sum of the areas of the rectangles with height M_i and width δx_i, giving an *upper* estimate for the area between the graph and the x-axis, as illustrated in Figure 6(b).

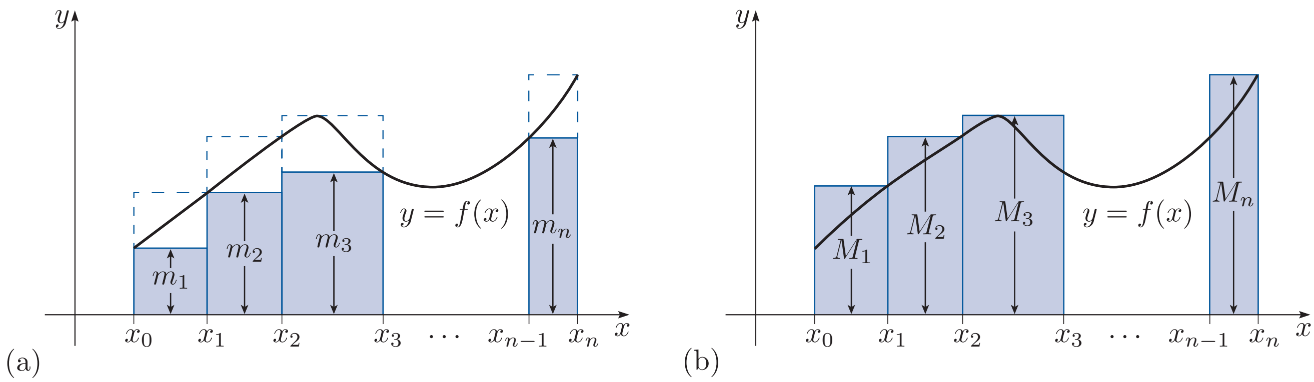


Figure 6 The rectangles whose areas are included in (a) the lower Riemann sum $L(f, P)$ and (b) the upper Riemann sum $U(f, P)$

Now, on any interval, the greatest lower bound of a function f is necessarily less than or equal to its least upper bound. It follows that, in each subinterval $[x_{i-1}, x_i]$, we have $m_i \leq M_i$. Summing from $i = 1$ to n, we obtain the following result.

Theorem F43

Let f be a bounded function on $[a, b]$, and let P be a partition of $[a, b]$. Then

$$L(f, P) \leq U(f, P).$$

Worked Exercise F26

Let

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 1, & x = 0, 1, \end{cases}$$

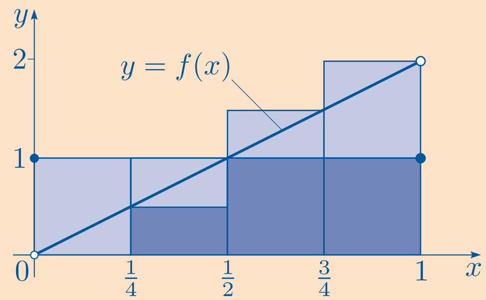
and let

$$P = \left\{ \left[0, \tfrac{1}{4}\right], \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \left[\tfrac{3}{4}, 1\right] \right\}$$

be a partition of $[0, 1]$. Determine $L(f, P)$ and $U(f, P)$.

Solution

It is helpful to make a sketch of the graph of f and the partition of the interval $[0,1]$.



Using the notation in the definitions, we set out the information needed to calculate the lower and upper Riemann sums using a layout that will make the calculation straightforward. Note that this function is increasing except at $x = 0$ and at $x = 1$, so we have to take special care at these points: the values of M_1 and m_4 are not what you might expect, and the values of m_1 and M_4 are not taken by the function.

For the four subintervals in P, we have

$$\begin{aligned}
&m_1 = 0, && M_1 = f(0) = 1, && \delta x_1 = \tfrac{1}{4},\\
&m_2 = f\left(\tfrac{1}{4}\right) = \tfrac{1}{2}, && M_2 = f\left(\tfrac{1}{2}\right) = 1, && \delta x_2 = \tfrac{1}{4},\\
&m_3 = f\left(\tfrac{1}{2}\right) = 1, && M_3 = f\left(\tfrac{3}{4}\right) = \tfrac{3}{2}, && \delta x_3 = \tfrac{1}{4},\\
&m_4 = f(1) = 1, && M_4 = 2, && \delta x_4 = \tfrac{1}{4}.
\end{aligned}$$

Then

$$\begin{aligned}
L(f,P) &= \sum_{i=1}^{4} m_i \delta x_i\\
&= \left(0 \times \tfrac{1}{4}\right) + \left(\tfrac{1}{2} \times \tfrac{1}{4}\right) + \left(1 \times \tfrac{1}{4}\right) + \left(1 \times \tfrac{1}{4}\right)\\
&= 0 + \tfrac{1}{8} + \tfrac{1}{4} + \tfrac{1}{4}\\
&= \tfrac{5}{8}
\end{aligned}$$

and

$$\begin{aligned}
U(f,P) &= \sum_{i=1}^{4} M_i \delta x_i\\
&= \left(1 \times \tfrac{1}{4}\right) + \left(1 \times \tfrac{1}{4}\right) + \left(\tfrac{3}{2} \times \tfrac{1}{4}\right) + \left(2 \times \tfrac{1}{4}\right)\\
&= \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{3}{8} + \tfrac{1}{2}\\
&= \tfrac{11}{8}.
\end{aligned}$$

Exercise F35

Let

$$f(x) = \begin{cases} 2x, & 0 < x < 1, \\ 1, & x = 0, 1, \end{cases}$$

and let

$$P = \left\{ \left[0, \tfrac{1}{5}\right], \left[\tfrac{1}{5}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, 1\right] \right\}$$

be a partition of $[0, 1]$. Determine $L(f, P)$ and $U(f, P)$.

Worked Exercise F26 and Exercise F35 addressed the same function f on the same interval $[0, 1]$ but with different partitions. The two lower sums were $5/8$ and $31/50$, and the two upper sums were $11/8$ and $3/2$; each of the two lower sums is smaller than *both* of the two upper sums.

We now look at another example to see whether this happens again in a different situation. We make use of the following formula for a sum of squares which you may have met in your previous studies (it can be proved by mathematical induction):

$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{for } n = 1, 2, \ldots.$$

Worked Exercise F27

Let

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1, \\ 2, & 1 < x \le 2, \end{cases}$$

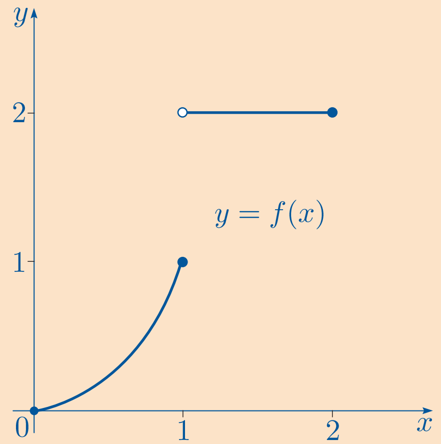
and, for each $n \in \mathbb{N}$, let

$$P_{2n} = \left\{ \left[0, \tfrac{1}{n}\right], \left[\tfrac{1}{n}, \tfrac{2}{n}\right], \ldots, \left[2 - \tfrac{1}{n}, 2\right] \right\}$$

be the standard partition of $[0, 2]$ into $2n$ equal subintervals. Determine $L(f, P_{2n})$ and $U(f, P_{2n})$.

Solution

The graph of f is shown below.



The function f is increasing on $[0,2]$. Thus, on each subinterval in $[0,2]$, the infimum of f is the value of f at the left endpoint of the subinterval and the supremum of f is the value of f at the right endpoint of the subinterval. Since the ith subinterval in P_{2n} is $[x_{i-1}, x_i] = \left[\frac{i-1}{n}, \frac{i}{n}\right]$, we have, for $i = 1, 2, \ldots, 2n$,

$$m_i = f\left(\frac{i-1}{n}\right), \quad M_i = f\left(\frac{i}{n}\right), \quad \delta x_i = \frac{i}{n} - \frac{i-1}{n} = \frac{1}{n}.$$

Now $f(x) = x^2$ on $[0,1]$ and $f(x) = 2$ on $(1,2]$ so, since $x_n = 1$, we have

$$m_i = \left(\frac{i-1}{n}\right)^2, \quad \text{for } i = 1, 2, \ldots, n+1,$$

and

$$m_i = 2, \quad \text{for } i = n+2, n+3, \ldots, 2n.$$

Hence

$$\begin{aligned} L(f, P_{2n}) &= \sum_{i=1}^{2n} m_i \delta x_i = \sum_{i=1}^{n+1} m_i \delta x_i + \sum_{i=n+2}^{2n} m_i \delta x_i \\ &= \left(0 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right) \times \frac{1}{n} + ((n-1) \times 2) \times \frac{1}{n} \\ &= \left(\frac{1^2 + \cdots + n^2}{n^2} \times \frac{1}{n}\right) + 2\frac{n-1}{n} \\ &= \frac{n(n+1)(2n+1)}{6n^3} + 2 - \frac{2}{n}. \end{aligned}$$

Also, we have

$$M_i = \left(\frac{i}{n}\right)^2, \quad \text{for } i = 1, 2, \ldots, n,$$

and

$$M_i = 2, \quad \text{for } i = n+1, n+2, \ldots, 2n.$$

Hence

$$\begin{aligned} U(f, P_{2n}) &= \sum_{i=1}^{2n} M_i \delta x_i \\ &= \sum_{i=1}^{n} M_i \delta x_i + \sum_{i=n+1}^{2n} M_i \delta x_i \\ &= \left(\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n}{n}\right)^2\right) \times \frac{1}{n} + (n \times 2) \times \frac{1}{n} \\ &= \left(\frac{1^2 + \ldots + n^2}{n^2} \times \frac{1}{n}\right) + 2 \\ &= \frac{n(n+1)(2n+1)}{6n^3} + 2. \end{aligned}$$

We now look at the result of Worked Exercise F27 for different values of n. As n increases, the number, $2n$, of subintervals in the partition increases and the length of each subinterval, $1/n$, decreases. From the above formulas for $L(f, P_{2n})$ and $U(f, P_{2n})$, we find that, to three decimal places, the lower and upper Riemann sums are then as given in the following table.

		$L(f, P_{2n})$	$U(f, P_{2n})$
$n = 2$:	4 equal subintervals	1.625	2.625
$n = 4$:	8 equal subintervals	1.969	2.469
$n = 10$:	20 equal subintervals	2.185	2.385
$n = 100$:	200 equal subintervals	2.318	2.338

As the subintervals increase in number and decrease in length, the lower sums increase and the upper sums decrease. But the lower sums are all less than or equal to all the upper sums! In fact this is always the case, as stated in the following result.

Theorem F44

Let f be a bounded function on $[a, b]$, and let P and P' be partitions of $[a, b]$. Then

$$L(f, P) \leq U(f, P').$$

You saw in Theorem F43 that, for a given partition P,

$$L(f, P) \leq U(f, P).$$

In order to prove the general case in Theorem F44 with two different partitions P and P' we require some new ideas. We develop these ideas and prove this result in Subsection 1.4.

The integral

We now return to our original problem: how to define an 'integral' that pins down our intuitive notion of 'the area under a curve'. We have seen that lower Riemann sums provide underestimates for this 'area' and upper Riemann sums provide overestimates; see Figure 7.

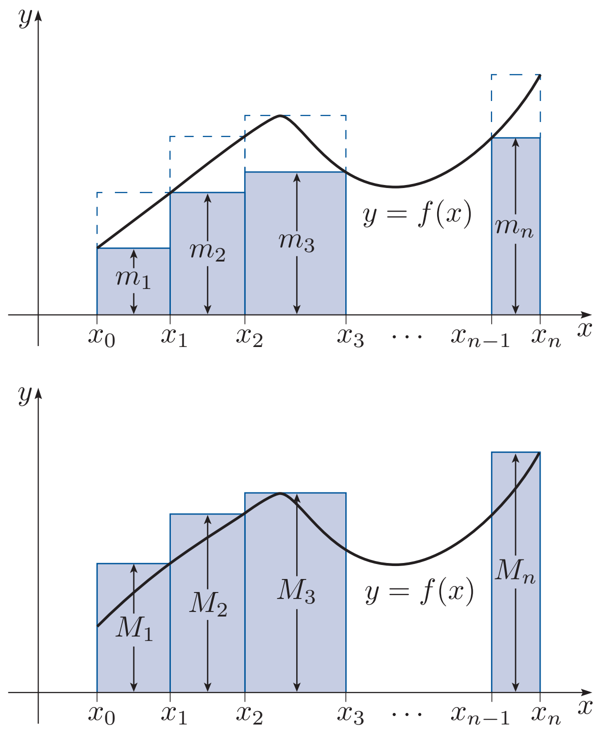


Figure 7 Lower and upper Riemann sums

So we make the following definitions.

Definitions

Let f be a bounded function on a closed interval $[a, b]$, let P be a partition of $[a, b]$ and let $L(f, P)$ and $U(f, P)$, respectively, be the corresponding lower and upper Riemann sums.

Then the **lower integral** of f on $[a, b]$ is

$$\underline{\int_a^b} f = \sup_P L(f, P),$$

and the **upper integral** of f on $[a, b]$ is

$$\overline{\int_a^b} f = \inf_P U(f, P).$$

We say that f is **integrable** on $[a, b]$ if

$$\underline{\int_a^b} f = \overline{\int_a^b} f,$$

and their common value is then called the **integral** of f on $[a, b]$.

The integral is written as $\int_a^b f$ or $\int_a^b f(x)\,dx$, and a and b are called the **limits of integration**.

Remarks

1. The common value of the lower and upper integrals of f (when it exists) is sometimes known as the *Riemann* integral of f, rather than simply as the integral of f.
2. In the definitions, $\sup_P L(f, P)$ is the supremum of the lower Riemann sums over *all possible* partitions P of the interval $[a, b]$. Similarly, $\inf_P U(f, P)$ is the infimum of the upper Riemann sums over all possible partitions P.
3. For any bounded function f on an interval $[a, b]$, it follows from the fact that all lower Riemann sums are less than or equal to all upper Riemann sums (Theorem F44), and from the above definitions, that the lower integral $\underline{\int_a^b} f$ and the upper integral $\overline{\int_a^b} f$ both exist, and that we always have

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(We omit the proofs of these facts.) Note, however, that the lower and upper integrals take different values unless f is integrable.

We now return to the function f that we considered earlier in Worked Exercise F27.

Worked Exercise F28

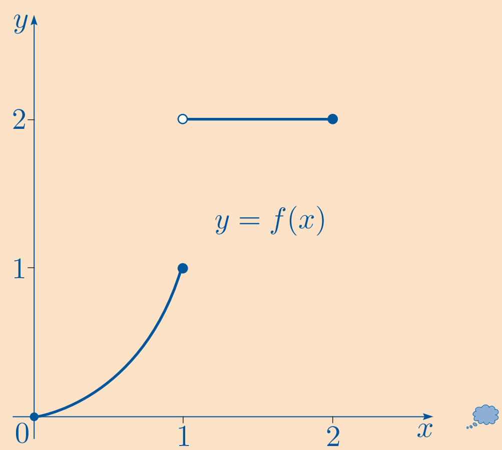
Let

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1, \\ 2, & 1 < x \le 2. \end{cases}$$

Prove that f is integrable on $[0,2]$, and evaluate $\int_0^2 f$.

Solution

It is helpful to sketch the graph.



We have already seen in Worked Exercise F27 that if we take P_{2n} to be the partition

$$\left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \dots, \left[2 - \frac{1}{n}, 2\right]\right\}$$

of $[0,2]$, then

$$L(f, P_{2n}) = \frac{n(n+1)(2n+1)}{6n^3} + 2 - \frac{2}{n}$$

and

$$U(f, P_{2n}) = \frac{n(n+1)(2n+1)}{6n^3} + 2.$$

Then, as $n \to \infty$, we have

$$L(f, P_{2n}) = \frac{(1+1/n)\,(2+1/n)}{6} + 2 - \frac{2}{n} \to \frac{1}{3} + 2 = \frac{7}{3}$$

so that, in particular,

$$\underline{\int_0^2} f \ge \frac{7}{3}.$$

This holds because $\underline{\int_0^2} f$ is defined to be the supremum of the lower Riemann sums of f over all possible partitions of $[0,2]$, so it must be greater than or equal to the particular lower Riemann sum $L(f, P_{2n})$ for any value of n.

Similarly, as $n \to \infty$, we have

$$U(f, P_{2n}) = \frac{(1+1/n)\,(2+1/n)}{6} + 2 \to \frac{1}{3} + 2 = \frac{7}{3}$$

so that

$$\overline{\int_0^2} f \leq \frac{7}{3}.$$

We have now shown that

$$\frac{7}{3} \leq \underline{\int_0^2} f \leq \overline{\int_0^2} f \leq \frac{7}{3}.$$

It follows that

$$\underline{\int_0^2} f = \overline{\int_0^2} f,$$

so f is integrable on $[0,2]$ and $\displaystyle\int_0^2 f = \frac{7}{3}$.

However, not all bounded functions defined on closed intervals are integrable!

Worked Exercise F29

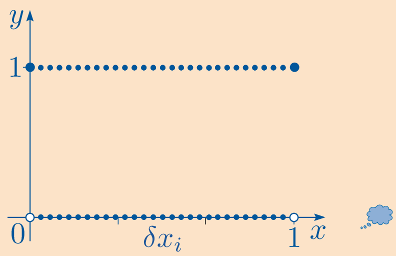
Let f be the Dirichlet function on $[0,1]$ defined by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1,\ x \text{ rational} \\ 0, & 0 \leq x \leq 1,\ x \text{ irrational}. \end{cases}$$

Determine the values of $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$, and hence show that f is not integrable on $[0,1]$.

Solution

The graph of f is shown below.



Let $P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$, where $x_0 = 0$, $x_1 = 1$, be any partition of $[0, 1]$. Then, on each subinterval $[x_{i-1}, x_i]$ in P, we have

$$m_i = 0 \text{ and } M_i = 1, \quad \text{for } i = 1, 2, \ldots, n.$$

This is because every subinterval contains both rational and irrational points, by the density property of the real numbers (see Subsection 1.4 of Unit D1).

So

$$L(f, P) = \sum_{i=1}^{n} m_i \delta x_i = \sum_{i=1}^{n} (0 \times \delta x_i) = 0$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \delta x_i = \sum_{i=1}^{n} (1 \times \delta x_i) = \sum_{i=1}^{n} \delta x_i = 1,$$

since the sum of the lengths of all the subintervals is equal to the length of the interval $[0, 1]$.

It follows that

$$\underline{\int_0^1} f = \sup_P L(f, P) = 0$$

and

$$\overline{\int_0^1} f = \inf_P U(f, P) = 1.$$

Then, since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$, we conclude that f is not integrable on $[0, 1]$.

Exercise F36

For the function

$$f(x) = x, \quad x \in [0, 1],$$

and the standard partition of $[0, 1]$

$$P_n = \left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right]\right\}, \quad n \in \mathbb{N},$$

determine the values of $L(f, P_n)$ and $U(f, P_n)$. Hence show whether f is integrable on $[0, 1]$ and, if it is, determine the value of $\int_0^1 f$.

1.2 Criteria for integrability

It would be tedious to have to go back to the definition of integrability whenever we wish to show that a given function is integrable on a closed interval. You will now meet a number of criteria that we can use to avoid this.

In order to prove that a bounded function f is integrable on $[a, b]$ directly from the definition of the integral, we need to look at $\sup_P L(f, P)$ and $\inf_P U(f, P)$ over *all* partitions P of $[a, b]$. However, as you saw in Worked Exercise F28 and Exercise F36, in many situations it is sufficient to consider just one *sequence* of partitions (P_n) in order to establish integrability. The conditions under which this simplification holds are set out in the following result. (Recall that $\|P_n\|$, the *mesh* of P_n, is the length of the longest subinterval of P_n.)

Theorem F45

Let f be a bounded function on $[a, b]$. If there is a sequence of partitions (P_n) of $[a, b]$ such that $\|P_n\| \to 0$ and

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = A, \quad \text{where } A \in \mathbb{R},$$

then f is integrable on $[a, b]$ and $\displaystyle\int_a^b f = A$.

Proof Let $\varepsilon > 0$. It follows from the equations in the statement of the theorem that there exists an integer n such that

$$L(f, P_n) > A - \tfrac{1}{2}\varepsilon \quad \text{and} \quad U(f, P_n) < A + \tfrac{1}{2}\varepsilon. \tag{1}$$

Now, by the definitions of upper and lower integrals,

$$\underline{\int_a^b} f \geq L(f, P_n) \quad \text{and} \quad \overline{\int_a^b} f \leq U(f, P_n). \tag{2}$$

Combining inequalities (1) and (2), we obtain

$$A - \tfrac{1}{2}\varepsilon < \underline{\int_a^b} f \le \overline{\int_a^b} f < A + \tfrac{1}{2}\varepsilon.$$

Here we have used the inequality $\underline{\int_a^b} f \le \overline{\int_a^b} f$ mentioned in the remarks after the definition of the integral in the previous subsection. As stated there, this follows from Theorem F44.

Since ε is any positive number, we deduce that the upper and lower integrals of f on $[a,b]$ are equal to A, so f is integrable and

$$\int_a^b f = A.$$

In fact the following result in the opposite direction to Theorem F45 also holds. However, its proof is somewhat more complicated, and we defer this to Subsection 1.4.

Theorem F46

If f is an integrable function on $[a,b]$ and (P_n) is a sequence of partitions of $[a,b]$ such that $\|P_n\| \to 0$, then

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

This result is particularly useful for proving that a function is *not* integrable. For if f is defined on $[a,b]$ and we can find a sequence of partitions (P_n) of $[a,b]$ whose mesh tends to zero but for which

$$\lim_{n\to\infty} L(f, P_n) \ne \lim_{n\to\infty} U(f, P_n),$$

then it follows from Theorem F46 that f is not integrable on $[a,b]$.

Exercise F37

For each of the following functions f, determine whether f is integrable on $[0,1]$ and, if it is, find $\int_0^1 f$.

(a) $f(x) = \begin{cases} -2, & 0 \le x < 1, \\ 3, & x = 1. \end{cases}$

(b) $f(x) = \begin{cases} x, & 0 \le x \le 1, \quad x \text{ rational}, \\ 0, & 0 \le x \le 1, \quad x \text{ irrational}. \end{cases}$

The next result is of particular interest in that its statement says nothing about the value of the integral itself: it mentions only the difference between the lower and the upper Riemann sums. The result follows from Theorems F45 and F46.

Corollary F47 Riemann's Criterion

Let f be bounded on $[a, b]$. Then

f is integrable on $[a, b]$

if and only if

there is a sequence (P_n) of partitions of $[a, b]$ with $\|P_n\| \to 0$ such that $U(f, P_n) - L(f, P_n) \to 0$.

Proof Theorem F46 implies that if f is integrable on $[a, b]$, and (P_n) is a sequence of partitions of $[a, b]$ with $\|P_n\| \to 0$, then

$$U(f, P_n) - L(f, P_n) \to 0.$$

On the other hand, if there is a sequence (P_n) of partitions of $[a, b]$ with $\|P_n\| \to 0$ such that $U(f, P_n) - L(f, P_n) \to 0$, then, because

$$L(f, P_n) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P_n),$$

these upper and lower integrals must be equal, so f is integrable on $[a, b]$, by Theorem F45. ■

Two classes of integrable functions

With Riemann's Criterion at our disposal, we can now determine some large classes of functions that are always integrable: the monotonic functions and the continuous functions.

Theorem F48

A function f which is bounded and monotonic on $[a, b]$ is integrable on $[a, b]$.

Proof We prove this theorem in the case when f is increasing on $[a, b]$. (The proof is similar if f is decreasing.)

Consider the standard partition of $[a, b]$; that is,

$$P_n = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}, \tag{3}$$

where

$$x_i = a + i\,\frac{b-a}{n}, \quad \text{for } i = 0, 1, 2, \ldots, n. \tag{4}$$

Now f is increasing, so on each subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$,

$$m_i = f(x_{i-1}) \quad \text{and} \quad M_i = f(x_i);$$

see Figure 8. Also, $\delta x_i = (b-a)/n$, for $i = 1, 2, \ldots, n$. Hence

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)\,\delta x_i \\ &= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\ &= \frac{b-a}{n} (f(x_n) - f(x_0)) \\ &= \frac{b-a}{n} (f(b) - f(a)). \end{aligned}$$

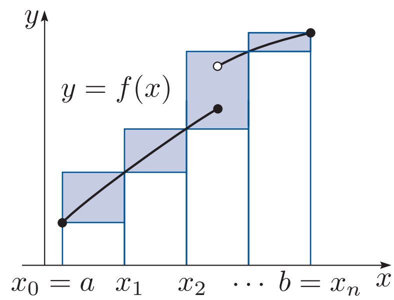


Figure 8 The standard partition for an increasing function f

Here we have used the fact that the terms of the series are

$$f(x_1) - f(x_0), \quad f(x_2) - f(x_1), \quad \ldots, \quad f(x_n) - f(x_{n-1}),$$

so everything except $f(x_0)$ and $f(x_n)$ cancels.

The sequence $((b-a)(f(b) - f(a))/n)$ is null, so it follows from Riemann's Criterion that f is integrable on $[a, b]$. ■

Theorem F49

A function f which is continuous on $[a, b]$ is integrable on $[a, b]$.

Proof We use the fact that f must be uniformly continuous on $[a, b]$.

You saw in Theorem F19 in Subsection 4.2 of Unit F1 that a function which is continuous on a bounded closed interval is uniformly continuous there.

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{b-a}, \quad \text{for all } x, y \in [a, b] \text{ with } |x - y| < \delta. \tag{5}$$

We use $\varepsilon/(b-a)$ here in order to obtain ε later in the proof.

Next we choose $N \in \mathbb{N}$ such that $(b-a)/N < \delta$. For $n \geq N$, let P_n be the standard partition of $[a, b]$ given by equations (3) and (4).

Now f is continuous on each subinterval $[x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$. Thus, by the Extreme Value Theorem (see Subsection 3.3 of Unit D4), there exist points c_i and d_i in $[x_{i-1}, x_i]$ such that

$$m_i = f(c_i) \quad \text{and} \quad M_i = f(d_i). \tag{6}$$

Since $[x_{i-1}, x_i]$ has length $(b-a)/n < \delta$, we deduce by statements (5) and (6) that

$$M_i - m_i < \frac{\varepsilon}{b-a}.$$

Here we have used the fact that $|d_i - c_i| < \dfrac{b-a}{n} < \delta$, from which it follows that $|f(d_i) - f(c_i)| = M_i - m_i < \dfrac{\varepsilon}{b-a}$ by statement (5).

Hence, for $n \geq N$ we have

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)\, \delta x_i \\ &< \sum_{i=1}^{n} \left(\frac{\varepsilon}{b-a}\right)\left(\frac{b-a}{n}\right) = \varepsilon. \end{aligned}$$

Thus $(U(f, P_n) - L(f, P_n))$ is a null sequence, so it follows from Riemann's Criterion that f is integrable on $[a, b]$. ■

Theorems F48 and F49 show that monotonic functions and continuous functions are integrable, but we know that some bounded functions are not integrable: for example, the Dirichlet function, as you saw in Worked Exercise F29. This suggests the question: precisely which bounded functions are integrable?

The full answer to this question is rather complicated but, roughly speaking, a bounded function is integrable on $[a, b]$ if and only if it is continuous at 'most' points of $[a, b]$. However, it is possible for a function to be discontinuous at infinitely many points of $[a, b]$ and yet be integrable on $[a, b]$. For example, the Riemann function which you met in Section 3 of Unit F1 is discontinuous at all rational points and yet it can be shown to be integrable on $[0, 1]$, the value of its integral being 0 (we do not prove this here).

Georg Friedrich Bernhard Riemann

Riemann and Lebesgue integration

Georg Friedrich Bernhard Riemann (1826–1866) laid down the fundamental ideas of the integral that is now named after him when he was writing his doctoral thesis in 1854. This was published posthumously in 1867. His formulation and proof were rather obscure and the version that is generally used today (including in this module) was given by the French mathematician Gaston Darboux (1842–1917) in 1875.

In 1902 a different definition of the integral was given by another French mathematician, Henri Léon Lebesgue (1875–1941). The main difference between the two definitions is in the way the area under the curve is measured. The Riemann integral considers the area as being made up of vertical rectangles, while the Lebesgue integral considers horizontal rectangles. Or to put it another way, the Riemann integral considers the domain of the function while the Lebesgue integral considers the codomain of the function. Although Lebesgue's definition has the advantage that it is applicable to a larger class of functions than Riemann's definition, it requires the formal notion of a measure. (You may study measure theory in the future, but this topic is not covered in M208.)

Lebesgue himself provided a rather nice example to illustrate the difference between his approach and that of Riemann:

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken out all my money I order the bills and coins according to identical values and then I pay the several heaps one after another to the creditor. This is my integral.

(Source: Siegmund-Schultze, R. (2008) 'Henri Lebesgue', in Gower, T. (ed) *The Princeton Companion to Mathematics*, Princeton, Princeton University Press, p. 796.)

Henri Léon Lebesgue

1.3 Properties of integrals

In your previous study of integration (for example, in a calculus course) you will have met many properties of integrals without a clear explanation of exactly why they hold. We now look at several of these properties and in some cases give an outline of how they follow from the definition of an integral in Subsection 1.1 and the various theorems that you met in Subsection 1.2. In reading this subsection, you should concentrate on understanding the various properties themselves; our comments on why the properties hold are optional reading in case you are interested.

First, we look at the limits of the integral. We have already defined integrals of the form $\displaystyle\int_a^b f$, where $a < b$; we now look at the situation where $a = b$ or $a > b$.

Definitions

Let f be a bounded function that is integrable on an interval I containing a and b, where $a < b$. Then we make the following definitions.

- $\displaystyle\int_a^a f = 0$
- $\displaystyle\int_b^a f = -\int_a^b f$

Thus, for example, $\displaystyle\int_1^0 x\,dx$ is defined to equal $-\displaystyle\int_0^1 x\,dx$; you have already seen in Exercise F36 that $\displaystyle\int_0^1 x\,dx = \tfrac{1}{2}$, so we define the value of $\displaystyle\int_1^0 x\,dx$ to be $-\tfrac{1}{2}$.

Next we look at the integrability of a bounded function on intervals with endpoints a, b and c, irrespective of the order of these endpoints on the x-axis. Figure 9 illustrates two possibilities: $a < c < b$ and $a < b < c$. The following result applies whatever the order of a, b and c.

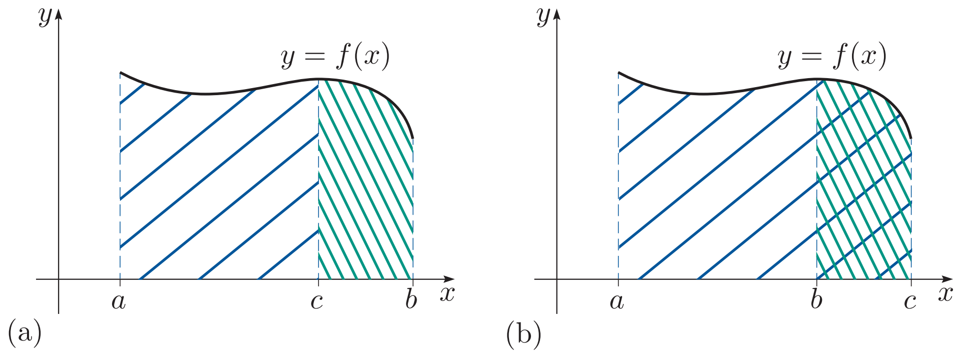


Figure 9 Integrating a function on intervals with endpoints a, b and c (a) when $a < c < b$ and (b) when $a < b < c$

Theorem F50 Additivity of integrals

Let f be a bounded function that is integrable on an interval I containing the points a, b and c. Then

$$\int_a^c f + \int_c^b f = \int_a^b f.$$

This result can be proved directly from the definitions above, and the definition of the integral given in Subsection 1.1. Notice that, in the situation illustrated in Figure 9(b), $\int_c^b f$ is negative since $b < c$.

Our next result says that the integral of a non-negative function is non-negative, and the integral of a non-positive function is non-positive; see Figure 10. This result can also be proved directly from the definition of the integral.

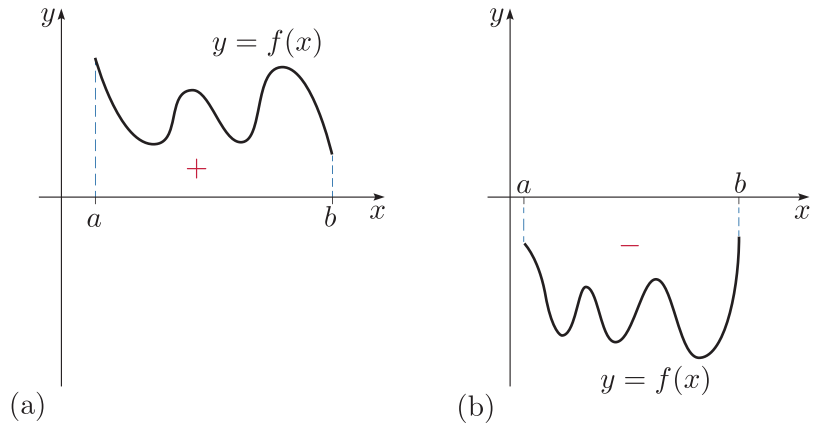


Figure 10 The sign of the integral for (a) a non-negative function and (b) a non-positive function

Theorem F51 Sign of an integral

Let f be a bounded function that is integrable on $[a, b]$.

- If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f \geq 0$.
- If $f(x) \leq 0$ on $[a, b]$, then $\displaystyle\int_a^b f \leq 0$.

Of course, a function can be non-negative on some parts of its domain and non-positive on other parts. In these circumstances, subintervals where the function takes negative values make a negative contribution to the total area between the graph of the function and the x-axis. If the interval $[a, b]$ on which a function is defined is made up of finitely many subintervals where the function is either always non-negative or always non-positive, then we can evaluate the integral $\displaystyle\int_a^b f$ by applying Theorem F50. If there are infinitely many such subintervals, then the evaluation of the integral involves summing an infinite series; we do not pursue this here.

Next we note that if a function f is integrable, then so is $|f|$; see Figure 11.

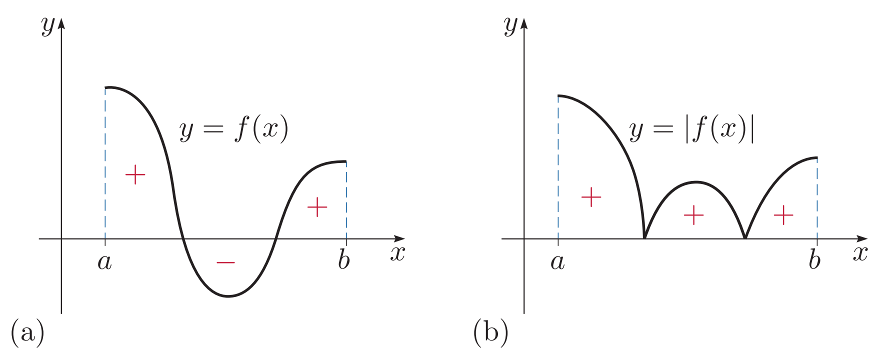


Figure 11 (a) The integral of f (b) The integral of $|f|$

Theorem F52 Modulus Rule

If f is integrable on $[a, b]$, then $|f|$ is also integrable on $[a, b]$.

Outline of the proof (optional)

For a bounded function f on $[a, b]$ and a partition of $[a, b]$

$$P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\},$$

we define, for $i = 1, 2, \ldots, n$, the **variation** $\omega_i(f)$ of f over the subinterval $[x_{i-1}, x_i]$ to be

$$\omega_i(f) = \sup\{|f(x) - f(y)| : x, y \in [x_{i-1}, x_i]\}.$$

It can be shown that

$$\omega_i(f) = \sup\{f(x) : x \in [x_{i-1}, x_i]\} - \inf\{f(x) : x \in [x_{i-1}, x_i]\}.$$

(We omit the details here.)

Hence

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} \omega_i(f)\delta x_i, \tag{7}$$

where $\delta x_i = x_i - x_{i-1}$, as usual.

Now, by the backwards form of the Triangle Inequality which you met in Subsection 3.1 of Unit D1, we have

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|, \quad \text{for } x, y \in [x_{i-1}, x_i],$$

so that

$$\omega_i(|f|) \leq \omega_i(f), \quad \text{for } i = 1, 2, \ldots, n.$$

Hence, by equation (7),

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P), \tag{8}$$

for any partition P of $[a, b]$. We can now use Riemann's Criterion to deduce from inequality (8) that if f is integrable on $[a, b]$, then so is $|f|$. ■

Finally, we set out the Combination Rules for integrable functions; we use these a great deal to construct 'new integrable functions from old'.

Theorem F53 Combination Rules for integrable functions

If f and g are integrable on $[a,b]$, then so are the following functions.

Sum Rule $f+g$, with integral

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Multiple Rule λf, for $\lambda \in \mathbb{R}$, with integral

$$\int_a^b \lambda f = \lambda \int_a^b f$$

Product Rule fg

Quotient Rule f/g, provided that $1/g$ is bounded on $[a,b]$.

You will meet some techniques for finding the integrals of products and quotients in the next section.

The Combination Rules can be proved using the same approach as for the Modulus Rule. The proof uses the following inequalities, which relate the variations of the new functions over a subinterval $[x_{i-1}, x_i]$ of a partition to those of the known integrable functions f and g:

- $\omega_i(f+g) \leq \omega_i(f) + w_i(g)$
- $\omega_i(\lambda f) \leq |\lambda| \omega_i(f), \text{ for } \lambda \in \mathbb{R}$
- $\omega_i(fg) \leq M(\omega_i(f) + \omega_i(g)), \text{ where } M = \max\{\sup|f|, \sup|g|\}.$

1.4 Proofs of Theorem F44 and Theorem F46 (optional)

In the proofs of these results, we will use the notion of a *refinement* of a partition. If P is a partition of an interval $[a,b]$, then any partition obtained from P by adding to it a finite number of partition points is called a **refinement** of P. The partition of $[a,b]$ obtained from two partitions P and P' of $[a,b]$ by using all their partition points is called the **common refinement** of P and P'.

For example, for the partitions

$$P = \{[0, \tfrac{1}{4}], [\tfrac{1}{4}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{3}{4}], [\tfrac{3}{4}, 1]\}$$

and

$$P' = \{[0, \tfrac{1}{3}], [\tfrac{1}{3}, \tfrac{2}{3}], [\tfrac{2}{3}, 1]\}$$

of $[0, 1]$, the common refinement of P and P' is the partition

$$\{[0, \tfrac{1}{4}], [\tfrac{1}{4}, \tfrac{1}{3}], [\tfrac{1}{3}, \tfrac{1}{2}], [\tfrac{1}{2}, \tfrac{2}{3}], [\tfrac{2}{3}, \tfrac{3}{4}], [\tfrac{3}{4}, 1]\},$$

as illustrated in Figure 12.

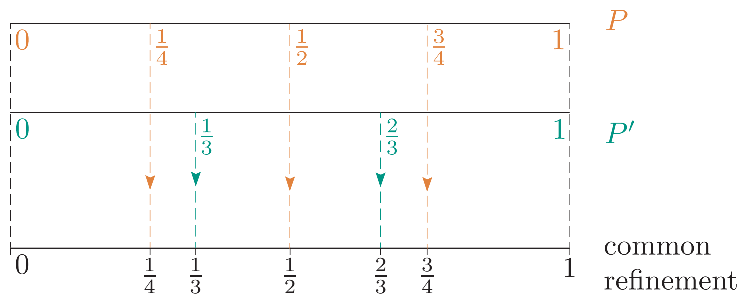


Figure 12 The common refinement of two partitions of $[0, 1]$

We first prove Theorem F44, which is restated below.

Theorem F44

Let f be a bounded function on $[a, b]$, and let P and P' be partitions of $[a, b]$. Then

$$L(f, P) \leq U(f, P').$$

Proof In our proof we will assume that f is non-negative on $[a, b]$. The general result can be deduced by applying the 'non-negative version' of the result to the function $g = f + c$, where c is a constant so large that g is non-negative on $[a, b]$; such a constant c exists since f is bounded on $[a, b]$.

First recall that, from Theorem F43, for any partition P of $[a, b]$ we have

$$L(f, P) \leq U(f, P).$$

We will prove Theorem F44 by showing that if P'' is the common refinement of P and P', then

$$L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P'). \tag{9}$$

We claim that adding a new partition point x' to a partition

$$P = \{[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\}$$

of $[a, b]$ does not increase the upper Riemann sum and may decrease it. This is because the new point x' lies in a subinterval $[x_{i-1}, x_i]$, for some i, and the only effect on the upper Riemann sum of adding x' is to replace the rectangle with side M_i standing on $[x_{i-1}, x_i]$ with a pair of adjacent rectangles standing on $[x_{i-1}, x_i]$ with heights at most M_i, as shown in Figure 13.

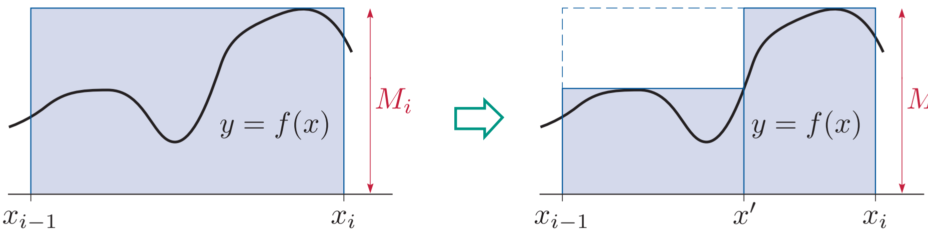


Figure 13 The effect on the upper Riemann sum of adding a new partition point

Hence adding a finite number of new partition points to P does not increase the upper Riemann sum. Similarly, adding a finite number of new partition points to P does not decrease the lower Riemann sum.

Now the partition P'' can be formed from either P or P' by adding a finite number of partition points, so inequalities (9) follow. ■

To end this subsection we give the proof of Theorem F46.

Theorem F46

If f is an integrable function on $[a, b]$ and (P_n) is a sequence of partitions of $[a, b]$ such that $\|P_n\| \to 0$, then

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = \int_a^b f.$$

Proof We prove that $\lim_{n\to\infty} U(f, P_n) = \int_a^b f$. The proof that $\lim_{n\to\infty} L(f, P_n) = \int_a^b f$ is similar.

We assume that f is non-negative on $[a, b]$ so, for some $M \in \mathbb{R}$, we have

$$0 \le f(x) \le M, \quad \text{for } a \le x \le b. \tag{10}$$

Let $\varepsilon > 0$. Since f is integrable on $[a, b]$, there is a partition

$$P' = \{[x'_0, x'_1], [x'_1, x'_2], \ldots, [x'_{m-1}, x'_m]\}$$

of $[a, b]$ where $x'_0 = a$ and $x'_m = b$, with m subintervals, such that

$$U(f, P') < \int_a^b f + \tfrac{1}{2}\varepsilon. \tag{11}$$

We use $\frac{1}{2}\varepsilon$ here in order to obtain ε later in the proof.

Now consider any partition in the sequence (P_n), of the form

$$P_n = \{[x_0, x_1], [x_1, x_2], \dots, [x_{p-1}, x_p]\},$$

where p is the number of subintervals in P_n. For $k = 1, 2, \dots, p$, we define

$$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\} \quad \text{and} \quad \delta x_k = x_k - x_{k-1}.$$

Let P'_n denote the common refinement of P' and P_n. Then, as in the proof of Theorem F44, we have

$$U(f, P'_n) \le U(f, P'). \tag{12}$$

Now we can obtain P_n from P'_n by removing at most $m - 1$ of the partition points of P'.

This is because P' has $m + 1$ partition points, and all partitions have the points a and b in common.

Removing such a point x'_i, lying in (x_{k-1}, x_k) say, we increase the upper Riemann sum by at most $M_k(x_k - x_{k-1})$, as illustrated in Figure 14. So, since $M_k \le M$, by inequality (10), and $x_k - x_{k-1} \le \|P_n\|$, for $k = 1, 2, \dots, p$,

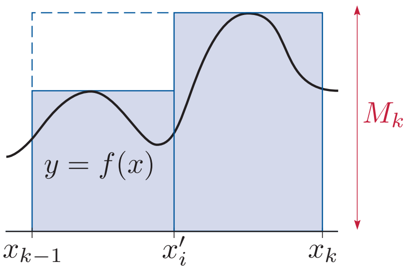


Figure 14 Removing a point x'_i of a partition

$$U(f, P_n) \le U(f, P'_n) + (m - 1)M\|P_n\|. \tag{13}$$

Combining inequalities (11), (12) and (13), we obtain

$$U(f, P_n) < \left(\int_a^b f + \tfrac{1}{2}\varepsilon\right) + (m - 1)M\|P_n\|.$$

Since $\|P_n\| \to 0$ as $n \to \infty$, we can choose N so large that

$$(m - 1)M\|P_n\| < \tfrac{1}{2}\varepsilon, \quad \text{for all } n > N.$$

We use $\frac{1}{2}\varepsilon$ here in order to obtain ε in the next step of the proof.

Hence

$$U(f, P_n) < \left(\int_a^b f + \tfrac{1}{2}\varepsilon\right) + \tfrac{1}{2}\varepsilon = \int_a^b f + \varepsilon, \quad \text{for all } n > N.$$

Since $U(f, P_n) \ge \displaystyle\int_a^b f$, by the definition of the integral, we deduce that

$$\left|U(f, P_n) - \int_a^b f\right| = U(f, P_n) - \int_a^b f < \varepsilon, \quad \text{for all } n > N.$$

So $U(f, P_n) - \displaystyle\int_a^b f$ tends to zero as n tends to ∞.

Hence $\displaystyle\lim_{n\to\infty} U(f, P_n) = \int_a^b f$, as required. ■

2 Evaluation of integrals

In Section 1 you saw what it means for a bounded function defined on a closed interval to be integrable. In this section you will study a variety of techniques for evaluating the integrals of such functions.

2.1 The Fundamental Theorem of Calculus

In this subsection we show that integration and differentiation are intimately related by proving a result known as the Fundamental Theorem of Calculus. You probably know this result from a previous course on calculus, but it is worth pausing to reflect on how remarkable it is. In Section 1 we defined the integral by means of more and more accurate estimates of the area between the graph of a function and the x-axis. At no point did it seem that this process was related to differentiation, but the Fundamental Theorem of Calculus shows that integration and differentiation are in some sense inverse processes. As you will see, this fact is enormously helpful in evaluating integrals.

We begin our exploration of these ideas by defining a *primitive* of a function.

Definition

Let f be a function defined on an interval I. Then a function F is a **primitive** of f on I if F is differentiable on I and

$$F'(x) = f(x), \quad \text{for } x \in I.$$

It follows from this definition that finding a primitive is the inverse of finding a derivative. (For this reason, a primitive is sometimes called an *antiderivative*.) Note that the domain of the primitive F may be larger than the interval I on which f is defined.

As an example of finding a primitive, let

$$f(x) = \tan x.$$

Then the function

$$F(x) = \log(\sec x)$$

is a primitive of f on the interval $(-\pi/2, \pi/2)$, since

$$F'(x) = \frac{1}{\sec x} \sec x \tan x = \tan x, \quad \text{for } x \in (-\pi/2, \pi/2).$$

Exercise F38

(a) Let

$$f(x) = (x^2 - 4)^{-1/2} \quad (x \in (2, \infty)).$$

Prove that

$$F(x) = \log\left(x + (x^2 - 4)^{1/2}\right)$$

is a primitive of f on $(2, \infty)$.

(b) Let

$$f(x) = \operatorname{sech} x \left(= \frac{1}{\cosh x}\right).$$

Prove that

$$F(x) = \tan^{-1}(\sinh x)$$

is a primitive of f on $\mathbb{R}$.

We now state and prove our main result, the Fundamental Theorem of Calculus. The theorem tells us that we can evaluate the integral of a function f on an interval $[a, b]$ by finding a primitive F of f on $[a, b]$. Note that the expression $F(b) - F(a)$ is sometimes written as $\big[F(x)\big]_a^b$ or $F(x)\big|_a^b$.

Theorem F54 Fundamental Theorem of Calculus

Let f be integrable on $[a, b]$ and let F be a primitive of f on $[a, b]$. Then

$$\int_a^b f = F(b) - F(a).$$

Proof Let

$$P_n = \{[x_0, x_1], \ldots, [x_{i-1}, x_i], \ldots, [x_{n-1}, x_n]\}, \quad \text{for } n = 1, 2, \ldots,$$

be a sequence of partitions of $[a, b]$, with $x_0 = a$, $x_n = b$ and $\|P_n\| \to 0$. On each subinterval

$$[x_{i-1}, x_i], \quad \text{for } i = 1, 2, \ldots, n,$$

the function F satisfies the conditions of the Mean Value Theorem which you met in Subsection 4.1 of Unit F2 *Differentiation*, since a primitive is differentiable and hence continuous. Thus there exists a point $c_i \in (x_{i-1}, x_i)$ such that

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(c_i)(x_i - x_{i-1}) \\ &= f(c_i)\,\delta x_i, \end{aligned} \tag{14}$$

where $\delta x_i = x_i - x_{i-1}$.

Now recall that we use the notation $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$ and $M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\}$.

Since

$$m_i \leq f(c_i) \leq M_i, \quad \text{for } i = 1, 2, \ldots, n,$$

it follows that

$$\sum_{i=1}^{n} m_i \, \delta x_i \leq \sum_{i=1}^{n} f(c_i) \, \delta x_i \leq \sum_{i=1}^{n} M_i \, \delta x_i.$$

Using equation (14), we can rewrite this statement as

$$L(f, P_n) \leq \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \leq U(f, P_n).$$

We now use telescopic cancellation to evaluate the series in these inequalities, which is

$$(F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \cdots + (F(x_n) - F(x_{n-1})).$$

The only remaining terms after cancellation are $F(x_n)$ and $-F(x_0)$.

The series has sum $F(x_n) - F(x_0) = F(b) - F(a)$, so

$$L(f, P_n) \leq F(b) - F(a) \leq U(f, P_n). \tag{15}$$

Since f is integrable on $[a, b]$, the sequences $(L(f, P_n))$ and $(U(f, P_n))$ both converge to $\int_a^b f$, by Theorem F46. It follows from inequalities (15) and the Limit Inequality Rule for sequences (Theorem D11 in Unit D2 *Sequences*) that

$$\int_a^b f \leq F(b) - F(a) \leq \int_a^b f,$$

which gives the required result. ■

Theorem F54 shows the close relationship between the integral $\int_a^b f$ and any primitive F of f on $[a, b]$. Because of this result, a primitive F is also called an **indefinite integral** of f and denoted by $\int f(x)\, dx$. Moreover, the process of finding a primitive of f is often informally called *integrating* f, and in this context the function f is called an **integrand**.

Also, the integral $\int_a^b f$ is often referred to as the **definite integral** of f over $[a, b]$.

We can use Theorem F54 and the table of standard primitives at the end of this unit to evaluate many integrals.

Worked Exercise F30

Evaluate $\displaystyle\int_0^1 2^x\,dx$.

Solution

Here the integrand is the function $x \longmapsto 2^x$; see the table at the end of this unit.

The function $f(x) = 2^x$ has the following primitive on $[0, 1]$:

$$F(x) = 2^x / \log 2.$$

Hence, by the Fundamental Theorem of Calculus,

$$\int_0^1 2^x\,dx = \left[\frac{2^x}{\log 2}\right]_0^1 = \frac{2}{\log 2} - \frac{1}{\log 2} = \frac{1}{\log 2}.$$

Exercise F39

Using the Fundamental Theorem of Calculus and the table of standard primitives, evaluate the following integrals.

(a) $\displaystyle\int_0^4 (x^2+9)^{1/2}\,dx$ (b) $\displaystyle\int_1^e \log x\,dx$

2.2 Primitives

It is natural to ask: can a function have more than one primitive on an interval? The answer to this question is 'yes': for example, on $(-1, 1)$ the functions

$$x \longmapsto x^2 \quad\text{and}\quad x \longmapsto x^2 + 1$$

are both primitives of the function

$$x \longmapsto 2x.$$

However, any two primitives of a function f on an interval can differ only by a constant.

Theorem F55 Uniqueness Theorem for Primitives

Let F_1 and F_2 be primitives of f on an interval I. Then there exists some constant c such that

$$F_2(x) = F_1(x) + c, \quad \text{for } x \in I.$$

Proof Since F_1 and F_2 are primitives of f on I,

$$F_1'(x) = f(x) \quad \text{and} \quad F_2'(x) = f(x), \quad \text{for } x \in I,$$

so

$$F_2'(x) - F_1'(x) = 0, \quad \text{for } x \in I.$$

Thus, by the Zero Derivative Theorem (Corollary F39 in Unit F2), there exists a constant c such that

$$F_2(x) - F_1(x) = c, \quad \text{for } x \in I. \qquad \blacksquare$$

The range of primitives we can find is considerably extended by the use of several Combination Rules. These rules can be proved using the corresponding rules for derivatives; we omit the details.

Theorem F56 Combination Rules for primitives

Let F and G be primitives of f and g, respectively, on an interval I, and let $\lambda \in \mathbb{R}$. Then, on I:

Sum Rule $f + g$ has a primitive $F + G$

Multiple Rule λf has a primitive λF

Scaling Rule $x \longmapsto f(\lambda x)$ has a primitive $x \longmapsto \dfrac{1}{\lambda} F(\lambda x)$, for $\lambda \neq 0$.

For example, it follows from the table of standard primitives and the Combination Rules that the function with domain $\mathbb{R}^+$ and rule

$$x \longmapsto 3x^{-1} + \sinh 2x$$

has a primitive

$$x \longmapsto 3 \log x + \tfrac{1}{2} \cosh 2x.$$

In applications of these Combination Rules we do not usually mention the rules explicitly.

Exercise F40

Using the table of standard primitives and the Combination Rules, find a primitive of each of the following functions.

(a) $f(x) = 4 \log x - 2/(4 + x^2) \quad (x \in (0, \infty))$

(b) $f(x) = 2 \tan 3x + e^{2x} \cos x \quad (x \in (-\pi/6, \pi/6))$

2.3 Techniques of integration

The Fundamental Theorem of Calculus provides a powerful method for evaluating certain integrals. However, even when we know that a function f has a primitive F, it may not be possible to determine F explicitly.

In fact, most functions f, even quite simple ones, have primitives which are not standard functions. For example, the primitives

$$\int e^{-x^2}\,dx \quad \text{and} \quad \int \frac{dx}{(\log x)^2}$$

cannot be expressed as a combination of a finite number of rational, nth root, trigonometric, exponential and logarithmic functions. (Note that here we have used a standard shorthand and written $\int \frac{dx}{(\log x)^2}$ instead of $\int \frac{1}{(\log x)^2}\,dx$.)

However, by using the Combination Rules we can certainly integrate any *polynomial* function, and the primitive is then always another polynomial function. For example,

$$\int (x^2 - x + 5)\,dx = \tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + 5x.$$

There is also is a standard procedure called the method of *partial fractions* for integrating a *rational* function. This procedure is not used in this module but is often used for evaluating the integrals of rational functions in complex analysis. Using this method, it can be shown that any primitive of a rational function can always be expressed in terms of rational functions, logarithms of rational functions and inverse tangents of linear functions.

There are also various techniques which can be applied to certain other types of function; the art of integration lies in recognising these types. We now describe briefly some basic techniques of integration.

Integration by substitution

We describe two related techniques of integration by substitution. The first is used when the integrand is of the form

$$x \longmapsto f(g(x))g'(x).$$

In this case, if F is a primitive of f, then

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x),$$

by the Composition Rule for derivatives (also called the Chain Rule). Thus $x \longmapsto F(g(x))$ is a primitive of $x \longmapsto f(g(x))g'(x)$ on any interval in the domain of $F \circ g$. So if we substitute $u = g(x)$, then we obtain a simpler integrand since

$$\int f(g(x))g'(x)\,dx = F(g(x)) = F(u) = \int f(u)\,du. \tag{16}$$

This technique is worth trying if you can express the integrand in the form $f(g(x))g'(x)$, for some functions f and g, as illustrated in the next worked exercise.

Worked Exercise F31

Find a primitive of the function

$$x \longmapsto x^2(x^3+1)^8 \quad (x \in \mathbb{R}).$$

Solution

Differentiating x^3+1 gives $3x^2$, which suggests that we try putting $u = g(x) = x^3+1$.

Put $u = g(x) = x^3 + 1$. Then $\dfrac{du}{dx} = g'(x) = 3x^2$, so $du = 3x^2\,dx$.

Now substitute to express the integral in terms of u, adjusting by a multiplicative constant as necessary.

We have

$$\begin{aligned}\int x^2(x^3+1)^8\,dx &= \tfrac{1}{3}\int (x^3+1)^8\,3x^2\,dx \\ &= \tfrac{1}{3}\int u^8\,du.\end{aligned}$$

Next, evaluate the integral in terms of u.

$$\begin{aligned}&= \tfrac{1}{3} \times \tfrac{1}{9}u^9 \\ &= \tfrac{1}{27}u^9\end{aligned}$$

Finally, substitute back for u to obtain the result in terms of x.

$$= \tfrac{1}{27}(x^3+1)^9.$$

Thus $\frac{1}{27}(x^3+1)^9$ is a primitive of $x^2(x^3+1)^8$.

You can always check your result by differentiating.

The following strategy summarises the approach taken in Worked Exercise F31.

Strategy F8

To find a primitive $\int f(g(x))g'(x)\,dx$ using integration by substitution, do the following.

1. Choose $u = g(x)$. Find $\dfrac{du}{dx} = g'(x)$ and hence express du in terms of x and dx.
2. Substitute $u = g(x)$ and replace $g'(x)\,dx$ by du (adjusting constants if necessary) to give $\int f(u)\,du$.
3. Find $\int f(u)\,du$.
4. Substitute $u = g(x)$ to give the required primitive in terms of x.

Remarks

1. If we are evaluating an integral, rather than finding a primitive, then there is no need to perform step 4 of Strategy F8. Instead, we can change the x-limits of integration into the corresponding u-limits:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

For example, in Worked Exercise F31 we have

$$u = g(x) = x^3 + 1,$$

so

$$\text{when } x = 0, \quad u = 1, \qquad \text{and when } x = 1, \quad u = 2.$$

Hence

$$\int_0^1 x^2(x^3+1)^8\,dx = \tfrac{1}{3}\int_1^2 u^8\,du.$$

2. When applying equation (16), you may be able to spot a primitive F of f immediately, in which case you can write down the required primitive $F(g(x))$ directly without need for a substitution. For example, if you were able to spot the primitive in Worked Exercise F31, you could just write

$$\begin{aligned}\int x^2(x^3+1)^8\,dx &= \tfrac{1}{3}\int (x^3+1)^8\,3x^2\,dx\\ &= \tfrac{1}{3}\times\tfrac{1}{9}(x^3+1)^9 = \tfrac{1}{27}(x^3+1)^9.\end{aligned}$$

One particular situation in which this simplification can often be applied is with an integrand of the form $g'(x)/g(x)$ on an interval I, since we have

$$\int \frac{g'(x)}{g(x)}\,dx = \log(g(x)), \quad \text{if } g(x) > 0, \quad \text{for } x \in I, \tag{17}$$

because $\dfrac{d}{dx}\log(g(x)) = \dfrac{1}{g(x)} \times g'(x)$.

An example involving this useful formula appears in the next exercise.

Exercise F41

Find a primitive of each of the following functions.

(a) $f(x) = \sin(\sin 3x)\cos 3x \quad (x \in \mathbb{R})$

(b) $f(x) = x^2(2+3x^3)^7 \quad (x \in \mathbb{R})$

(c) $f(x) = x\sin(2x^2) \quad (x \in \mathbb{R})$

(d) $f(x) = x/(2+3x^2) \quad (x \in \mathbb{R})$

Exercise F42

Evaluate the integral

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx.$$

Our second substitution technique is a modification of the above method, which we call *backwards substitution*. It is based on the formula

$$\int f(x)\,dx = \int f(h(u))h'(u)\,du,$$

obtained from equation (16) by swapping the variables x and u. Here h is a function such that $x = h(u)$, often found by first writing $u = g(x)$ where g has an inverse function $g^{-1} = h$. Backwards substitution is worth trying if it makes part of the integrand significantly simpler. The next worked exercise gives an example of the use of this technique.

Worked Exercise F32

Find a primitive of the function

$$x \longmapsto \frac{e^{2x}}{(e^x-1)^{1/2}} \quad (x \in (0,\infty)).$$

Solution

If $u = g(x) = (e^x-1)^{1/2}$, then g has an inverse function $x = h(u) = \log(u^2+1)$.

Put $u = g(x) = (e^x-1)^{1/2}$. Then the inverse function is $x = h(u) = \log(u^2+1)$, so $\dfrac{dx}{du} = h'(u) = \dfrac{2u}{u^2+1}$ and therefore $dx = \dfrac{2u}{u^2+1}\,du$.

Now substitute to express the integral in terms of u.

We have

$$\begin{aligned}\int \frac{e^{2x}}{(e^x-1)^{1/2}}\,dx &= \int \frac{e^{\log(u^2+1)^2}}{u}\,\frac{2u}{u^2+1}\,du\\ &= \int \frac{(u^2+1)^2}{u}\,\frac{2u}{u^2+1}\,du\\ &= \int 2(u^2+1)\,du\end{aligned}$$

Next, evaluate the integral in terms of u.

$$= \tfrac{2}{3}u^3 + 2u$$

Finally, substitute back for u to obtain the result in terms of x.

$$= \tfrac{2}{3}(e^x-1)^{3/2} + 2(e^x-1)^{1/2}.$$

Thus $\frac{2}{3}(e^x-1)^{3/2} + 2(e^x-1)^{1/2}$ is a primitive of $\dfrac{e^{2x}}{(e^x-1)^{1/2}}$.

The following strategy summarises the approach taken in Worked Exercise F32.

Strategy F9

To find a primitive $\int f(x)\,dx$ using integration by backwards substitution, do the following.

1. Choose $u = g(x)$, where g has an inverse function $x = h(u)$. Find $\dfrac{dx}{du} = h'(u)$ and hence express dx in terms of u and du.
2. Substitute $x = h(u)$ and replace dx by $h'(u)\,du$ to give a primitive in terms of u.
3. Find this primitive.
4. Substitute $u = g(x)$ to give the required primitive in terms of x.

As before, if we are evaluating an integral, then instead of step 4 in Strategy F9, we can change the x-limits of integration into the corresponding u-limits:

$$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(g^{-1}(u))(g^{-1})'(u)\,du.$$

For example, in Worked Exercise F32 we have

$$u = g(x) = (e^x - 1)^{1/2},$$

so

$$\text{when } x = \log 2, \quad u = 1, \text{ and when } x = \log 3, \quad u = \sqrt{2}.$$

Hence

$$\int_{\log 2}^{\log 3} \frac{e^{2x}}{(e^x - 1)^{1/2}}\,dx = \int_1^{\sqrt{2}} 2(u^2 + 1)\,du.$$

Exercise F43

(a) Find a primitive of the function

$$f(x) = \frac{1}{3(x-1)^{3/2} + x(x-1)^{1/2}} \quad (x \in (1, \infty)),$$

using the substitution $u = (x-1)^{1/2}$.

(b) Evaluate the integral

$$\int_0^{\log 3} e^x \sqrt{1 + e^x}\,dx.$$

Integration by parts

The technique of *integration by parts* is derived from the Product Rule for differentiation,

$$(fg)' = f'g + fg',$$

which implies that

$$\int fg' = fg - \int f'g, \quad \text{so} \quad \int_a^b fg' = [fg]_a^b - \int_a^b f'g.$$

This formula converts the problem of finding a primitive of fg' into the problem of finding a primitive of $f'g$. Integration by parts is worth trying if you can express the integrand as a product of two functions, $f(x)g'(x)$, where $f(x)$ becomes simpler on differentiation, and $g'(x)$ becomes not much more complicated on integration.

Here is a strategy for using integration by parts.

Strategy F10

To find a primitive $\int k(x)\,dx$ using integration by parts, do the following.

1. Write the original function k in the form fg', where f is a function that you can differentiate and g' is a function that you can integrate.
2. Use the formula $\int fg' = fg - \int f'g$.

The next two worked exercises give examples of using this technique.

Worked Exercise F33

Find

$$\int x \cos x\,dx.$$

Solution

We use integration by parts with $f(x) = x$ and $g'(x) = \cos x$.

Take $f(x) = x$ and $g'(x) = \cos x$, so that $f'(x) = 1$ and $g(x) = \sin x$. Then

$$\begin{aligned}\int x \cos x\,dx &= x \sin x - \int \sin x\,dx \\ &= x \sin x + \cos x.\end{aligned}$$

Sometimes we have to multiply the integrand by the factor 1 in order to be able to apply integration by parts, as in the following example.

Worked Exercise F34

Evaluate the integral

$$\int_0^1 \tan^{-1} x \, dx.$$

Solution

We use integration by parts, introducing the factor 1:

Here

$$f(x) = \tan^{-1} x, \quad g'(x) = 1,$$

so

$$f'(x) = \frac{1}{1+x^2}, \quad g(x) = x.$$

$$\begin{aligned}
\int_0^1 \tan^{-1} x \, dx &= \int_0^1 1 \times \tan^{-1} x \, dx \\
&= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \\
&= \tan^{-1} 1 - \left[\tfrac{1}{2}\log(1+x^2)\right]_0^1 \\
&= \frac{\pi}{4} - \left(\frac{1}{2}\log 2 - \frac{1}{2}\log 1\right) \\
&= \frac{\pi}{4} - \frac{1}{2}\log 2.
\end{aligned}$$

Here we have used equation (17) with $g(x) = 1 + x^2$.

Exercise F44

(a) Find a primitive of the function

$$k(x) = x^{1/3} \log x \quad (x \in \mathbb{R}^+).$$

(b) Evaluate the integral

$$\int_0^{\pi/2} x^2 \cos x \, dx.$$

Hint: Use integration by parts twice.

Reduction formulas

Sometimes we need to evaluate an integral I_n that involves a non-negative integer n. A common approach to such integrals is to relate the value of I_n to the value of I_{n-1} or I_{n-2} by a **reduction formula** (sometimes called a *recurrence relation*) using integration by parts. Here is an example that will be important later in the unit.

Worked Exercise F35

Let

$$I_n = \int_0^{\pi/2} \sin^n x\,dx, \quad n = 0, 1, 2, \ldots.$$

(a) Evaluate I_0 and I_1.

(b) Prove that

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}, \quad \text{for } n \geq 2.$$

(c) Deduce the values of I_2, I_3, I_4 and I_5.

Solution

(a) We have $I_0 = \displaystyle\int_0^{\pi/2} 1\,dx = \pi/2$ and

$$I_1 = \int_0^{\pi/2} \sin x\,dx = \big[-\cos x\big]_0^{\pi/2} = 1.$$

(b) We write

$$I_n = \int_0^{\pi/2} \sin x\,\sin^{n-1} x\,dx.$$

We integrate $\sin x$ and differentiate $\sin^{n-1} x$.

Using integration by parts we find that, for $n \geq 2$,

$$\begin{aligned}
I_n &= \big[(-\cos x)\sin^{n-1} x\big]_0^{\pi/2} - \\
&\qquad \int_0^{\pi/2} (-\cos x)(n-1)\sin^{n-2} x \cos x\,dx \\
&= 0 + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2} x\,dx \\
&= (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2} x\,dx \\
&= (n-1)\left(\int_0^{\pi/2} \sin^{n-2} x\,dx - \int_0^{\pi/2} \sin^n x\,dx\right) \\
&= (n-1)(I_{n-2} - I_n).
\end{aligned}$$

We can rearrange this equation to give

$$nI_n = (n-1)I_{n-2}, \quad \text{so} \quad I_n = \left(\frac{n-1}{n}\right) I_{n-2}, \quad \text{for } n \geq 2.$$

(c) Using the result of part (b) with $n = 2, 3, 4, 5$ in turn, we obtain

$$\begin{aligned} I_2 &= \frac{1}{2} I_0 = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}, \\ I_3 &= \frac{2}{3} I_1 = \frac{2}{3}, \\ I_4 &= \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{\pi}{4} = \frac{3\pi}{16}, \\ I_5 &= \frac{4}{5} I_3 = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}. \end{aligned}$$

By repeatedly applying the reduction formula in Worked Exercise F35, we obtain the general formulas

$$I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n} \cdot \frac{\pi}{2}$$

and

$$I_{2n+1} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n+1}.$$

We will use these formulas in Subsection 3.2.

Exercise F45

Let

$$I_n = \int_0^1 e^x x^n \, dx, \quad n = 0, 1, 2, \ldots.$$

(a) Evaluate I_0.

(b) Prove that

$$I_n = e - nI_{n-1}, \quad \text{for } n = 1, 2, \ldots.$$

(c) Deduce the values of I_1, I_2, I_3 and I_4.

3 Inequalities, sequences and series

Often it is not possible to evaluate an integral explicitly, and a numerical estimate for its value is sufficient. This situation can arise both in applications of mathematics and in proofs that involve integration. In this section we study some inequalities satisfied by integrals, and apply these to find two remarkable formulas for π, and to decide whether certain series are convergent or divergent.

3.1 Inequalities for integrals

The basic inequality rules for integrals are as follows.

Theorem F57 Inequality Rules

Let f and g be integrable on $[a, b]$.

(a) If $f(x) \leq g(x)$, for $x \in [a, b]$, then

$$\int_a^b f \leq \int_a^b g.$$

(b) If $m \leq f(x) \leq M$, for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

Proof The proofs of parts (a) and (b) of Theorem F57 are illustrated in Figure 15.

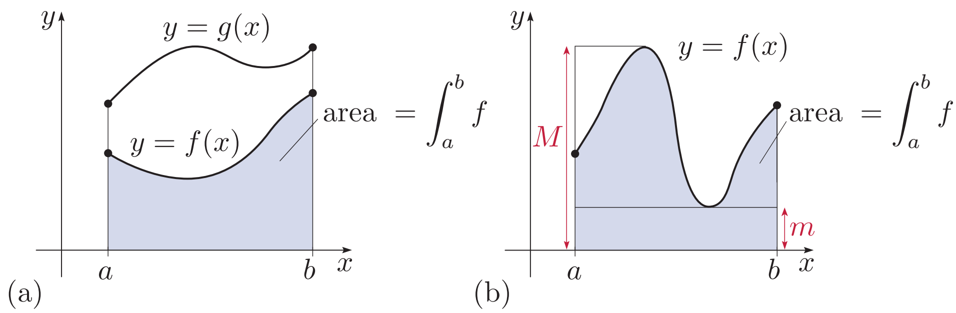


Figure 15 The proofs of the Inequality Rules parts (a) and (b)

(a) Let P be any partition of $[a, b]$. Since

$$f(x) \leq g(x), \quad \text{for } x \in [a, b],$$

the infimum of f on each subinterval of P is less than or equal to the infimum of g on that subinterval, and therefore $L(f, P) \leq L(g, P)$. Thus

$$\int_a^b f = \sup_P L(f, P) \leq \sup_P L(g, P) = \int_a^b g,$$

since f and g are both integrable on $[a, b]$.

(b) Since $f(x) \leq M$ for $x \in [a, b]$, it follows from part (a), with $g(x) = M$, that

$$\int_a^b f \leq \int_a^b M\,dx = M(b-a).$$

The proof of the left-hand inequality is similar. ■

The Inequality Rules allow us to estimate a complicated integral by evaluating a simpler one, as in the next worked exercise.

Worked Exercise F36

Prove the following inequalities.

(a) $\displaystyle\int_0^1 \frac{x^3}{2-\sin^4 x}\,dx \le \tfrac{1}{4}\log 2$ (b) $\displaystyle\frac{3}{\sqrt{34}} \le \int_{-1}^2 \frac{dx}{\sqrt{2+x^5}} \le 3$

Solution

(a) Since

$$|\sin x| \le |x|, \quad \text{for } x \in \mathbb{R},$$

This is Corollary D46 from Subsection 2.3 of Unit D4.

it follows that

$$\sin^4 x \le x^4, \quad \text{for } x \in \mathbb{R}.$$

Thus $2-\sin^4 x \ge 2 - x^4 > 0$ for $x \in [0,1]$, so

$$\frac{x^3}{2-\sin^4 x} \le \frac{x^3}{2-x^4}, \quad \text{for } x \in [0,1].$$

Hence, by Inequality Rule (a), we have

$$\begin{aligned}\int_0^1 \frac{x^3}{2-\sin^4 x}\,dx &\le \int_0^1 \frac{x^3}{2-x^4}\,dx\\ &= \left[-\tfrac{1}{4}\log(2-x^4)\right]_0^1\\ &= -\tfrac{1}{4}(\log 1 - \log 2)\\ &= \tfrac{1}{4}\log 2.\end{aligned}$$

Here we have used equation (17) with $g(x) = 2 - x^4$.

(b) Since the function $x \longmapsto \sqrt{2+x^5}$ is increasing on $[-1,2]$, we have

$$1 \le \sqrt{2+x^5} \le \sqrt{34}, \quad \text{for } x \in [-1,2],$$

so

$$\frac{1}{\sqrt{34}} \le \frac{1}{\sqrt{2+x^5}} \le 1, \quad \text{for } x \in [-1,2].$$

Since the length of the interval $[-1,2]$ is 3, it follows from Inequality Rule (b) that

$$\frac{3}{\sqrt{34}} \le \int_{-1}^2 \frac{dx}{\sqrt{2+x^5}} \le 3.$$

Exercise F46

Use the Inequality Rules to prove the following inequalities.

(a) $\displaystyle\int_1^3 x\sin\big(1/x^{10}\big)\,dx \le 4$ (b) $\frac{1}{2} \le \displaystyle\int_0^{1/2} e^{x^2}\,dx \le \frac{1}{2}e^{1/4}$

We saw in Subsection 1.3 that if the function f is integrable on $[a,b]$, then so is $|f|$. We now use the Inequality Rules to obtain an inequality involving the integrals of f and $|f|$, known as the Triangle Inequality for integrals. The name arises because of the similarity between this inequality and the Triangle Inequality for numbers:

$$\left|\sum_{i=1}^{n} a_i\right| \le \sum_{i=1}^{n} |a_i|.$$

Theorem F58 Triangle Inequality for integrals

Let f be integrable on $[a,b]$. Then

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Furthermore, if $|f(x)| \le M$ for $x \in [a,b]$, then

$$\left|\int_a^b f\right| \le M(b-a).$$

Proof We know that, for all $x \in [a,b]$,

$$-|f(x)| \le f(x) \le |f(x)|.$$

Since $|f|$ is integrable on $[a,b]$, it follows from Inequality Rule (a) that

$$-\int_a^b |f| \le \int_a^b f \le \int_a^b |f|,$$

which is equivalent to

$$\left|\int_a^b f\right| \le \int_a^b |f|.$$

Finally, if $|f(x)| \le M$ for $x \in [a,b]$, then, by the above inequality and Inequality Rule (b),

$$\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a).$$

■

Sometimes we use the Inequality Rules and the Triangle Inequality in combination, as in the next worked exercise.

Worked Exercise F37

Prove that

$$\left|\int_0^{\pi/2} \frac{x-\pi/2}{2+\cos x}\,dx\right| \leq \frac{\pi^2}{16}.$$

Solution

By the Triangle Inequality for integrals,

$$\begin{aligned}\left|\int_0^{\pi/2} \frac{x-\pi/2}{2+\cos x}\,dx\right| &\leq \int_0^{\pi/2} \left|\frac{x-\pi/2}{2+\cos x}\right| dx \\ &= \int_0^{\pi/2} \frac{\pi/2-x}{2+\cos x}\,dx. \qquad (*)\end{aligned}$$

For $0 \leq x \leq \pi/2$, we have $\pi/2 - x \geq 0$ and $\cos x \geq 0$.

Next, since

$$2+\cos x \geq 2, \quad \text{for } x \in [0, \pi/2],$$

we have

$$\frac{1}{2+\cos x} \leq \frac{1}{2}, \quad \text{for } x \in [0, \pi/2].$$

Thus, by Inequality Rule (a) and statement $(*)$,

$$\begin{aligned}\left|\int_0^{\pi/2} \frac{x-\pi/2}{2+\cos x}\,dx\right| &\leq \int_0^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2}-x\right) dx \\ &= \frac{1}{2}\left[\frac{\pi}{2}x - \frac{1}{2}x^2\right]_0^{\pi/2} \\ &= \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) \\ &= \frac{\pi^2}{16}.\end{aligned}$$

Exercise F47

Prove the following inequalities.

(a) $\displaystyle\left|\int_1^4 \frac{\sin(1/x)}{2+\cos(1/x)}\,dx\right| \leq 3$ (b) $\displaystyle\left|\int_0^{\pi/4} \frac{\tan x}{3-\sin(x^2)}\,dx\right| \leq \tfrac{1}{4}\log 2$

3.2 Wallis' Formula

In Worked Exercise F35 we used a reduction formula to show that if

$$I_n = \int_0^{\pi/2} \sin^n x\,dx, \quad n = 0, 1, 2, \ldots,$$

then

$$I_0 = \frac{\pi}{2}, \quad I_1 = 1 \quad \text{and} \quad I_n = \left(\frac{n-1}{n}\right) I_{n-2}, \quad \text{for } n \geq 2. \tag{18}$$

We also remarked that by repeatedly applying equations (18) we obtain

$$I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n} \cdot \frac{\pi}{2}, \quad \text{for } n \geq 1, \tag{19}$$

and

$$I_{2n+1} = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n+1}, \quad \text{for } n \geq 1. \tag{20}$$

Note that the formula for I_{2n} involves π, but the formula for I_{2n+1} does not.

We now use these results, together with inequalities between various integrals of the form I_n, to establish two remarkable formulas for π, the first of which is Wallis' Formula.

Theorem F59

(a) $$\lim_{n\to\infty} \left(\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{\pi}{2}$$

(b) $$\lim_{n\to\infty} \frac{(n!)^2\, 2^{2n}}{(2n)!\sqrt{n}} = \sqrt{\pi}$$

John Wallis

John Wallis (1616–1703) was the most influential English mathematician before the rise of Isaac Newton. He was a cryptographer to the Parliamentarians during the Civil War and in 1649 was appointed Savilian Professor of Geometry at Oxford.

His most important work is his *Arithmetica Infinitorum* (Arithmetic of Infinitesimals), published in 1656. It was in this work that Wallis derived the formula which bears his name, and it was through studying this work that Newton came to discover his version of the binomial theorem.

Wallis' treatise on conic sections, published in 1655, contains the first publication of the symbol for infinity, ∞.

Part (c) of the next exercise gives an identity needed in the proof of Theorem F59. If you are short of time, then you may wish to skip this part of the exercise, which is quite challenging, and skim read the solution and also the proof of the theorem.

Exercise F48

For $n = 1, 2, \dots,$ let

$$a_n = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \cdots \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \quad \text{and} \quad b_n = \frac{(n!)^2\, 2^{2n}}{(2n)!\sqrt{n}}.$$

(a) Evaluate a_n and b_n, for $n = 1, 2, 3$.

(b) Verify that

$$b_n^2 = \left(\frac{2n+1}{n}\right) a_n, \quad \text{for } n = 1, 2, 3.$$

(c) Prove that

$$b_n^2 = \left(\frac{2n+1}{n}\right) a_n, \quad \text{for } n = 1, 2, \dots.$$

Proof of Theorem F59

Let

$$I_n = \int_0^{\pi/2} \sin^n x\, dx, \quad n = 0, 1, 2, \dots,$$

and let the sequences (a_n) and (b_n) be as given in Exercise F48.

(a) Using equations (19) and (20), we obtain, for $n \geq 1$,

$$\frac{I_{2n}}{I_{2n+1}} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)(2n+1)}{2 \cdot 2 \cdot 4 \cdot 4 \cdot \cdots \cdot (2n)(2n)} \frac{\pi}{2} = \frac{1}{a_n} \cdot \frac{\pi}{2},$$

so

$$a_n = \left(\frac{I_{2n+1}}{I_{2n}}\right) \frac{\pi}{2}.$$

Thus to prove part (a), it is sufficient to show that

$$\frac{I_{2n+1}}{I_{2n}} \to 1 \text{ as } n \to \infty. \tag{21}$$

We do this as follows. Since $0 \leq \sin x \leq 1$, for $x \in [0, \pi/2]$, we have

$$\sin^{2n} x \geq \sin^{2n+1} x \geq \sin^{2n+2} x, \quad \text{for } x \in [0, \pi/2].$$

It follows by Inequality Rule (a) that

$$I_{2n} \geq I_{2n+1} \geq I_{2n+2}.$$

Thus, on dividing by I_{2n}, we obtain

$$1 \geq \frac{I_{2n+1}}{I_{2n}} \geq \frac{I_{2n+2}}{I_{2n}} = \frac{2n+1}{2n+2} = \frac{2+1/n}{2+2/n},$$

by equation (18). On taking the limit as $n \to \infty$, we deduce that statement (21) holds, by the Squeeze Rule for sequences from Unit D2.

(b) We know, from Exercise F48(c), that

$$b_n^2 = \left(\frac{2n+1}{n}\right) a_n = \left(2 + \frac{1}{n}\right) a_n.$$

By part (a),

$$a_n \to \frac{\pi}{2} \quad \text{as } n \to \infty,$$

so, by the Product Rule for sequences,

$$b_n^2 \to 2 \times \frac{\pi}{2} = \pi \quad \text{as } n \to \infty.$$

Hence, by the continuity of the square root function,

$$b_n \to \sqrt{\pi} \quad \text{as } n \to \infty.$$

■

3.3 The Integral Test

In this subsection we introduce a method based on integration for determining the convergence or divergence of certain series of the form $\sum_{n=1}^{\infty} f(n)$, where the function f is positive and decreasing and tends to 0. The method is based on the fact that it is often easier to evaluate an integral than a sum which has a similar behaviour.

Theorem F60 Integral Test

Let the function f be positive and decreasing on $[1, \infty)$, and suppose that $f(x) \to 0$ as $x \to \infty$. Then

(a) $\displaystyle\sum_{n=1}^{\infty} f(n)$ converges if the sequence $\left(\displaystyle\int_1^n f\right)$ is bounded above

(b) $\displaystyle\sum_{n=1}^{\infty} f(n)$ diverges if $\displaystyle\int_1^n f \to \infty$ as $n \to \infty$.

Remarks

1. The Integral Test is also called the *Maclaurin Integral Test*.
2. In both parts (a) and (b), the number 1 can be replaced by any positive integer.

Colin Maclaurin (1698–1746) was a Scottish mathematician who spent most of his career as professor of mathematics at Edinburgh University, having been appointed on the recommendation of Isaac Newton. A popular teacher, he was described as a 'favourite professor' and the 'life and soul' of the university. He is notable for having extended Newton's work on the calculus and geometry. His two-volume *Treatise of Fluxions* (1742), which was the first systematic treatment of Newton's methods, contains a detailed discussion of infinite series and includes the Integral Test in verbal form.

The Integral Test was rediscovered by Cauchy – he published it in 1827 – and consequently it is also known as the Maclaurin–Cauchy Integral Test.

Colin Maclaurin

Proof of Theorem F60 For $n = 2, 3, \dots$, let

$s_n = f(1) + f(2) + \cdots + f(n)$ be the nth partial sum of the series $\sum_{n=1}^{\infty} f(n)$, and let P_{n-1} be the standard partition of $[1, n]$ with $n - 1$ subintervals of length 1, that is,

$$\{[1,2], \dots, [i, i+1], \dots, [n-1, n]\}.$$

Since f is decreasing on $[1, \infty)$, we have, for $i = 1, 2, \dots, n-1$,

$$m_i = f(i+1) \quad \text{and} \quad M_i = f(i).$$

This is illustrated in Figure 16.

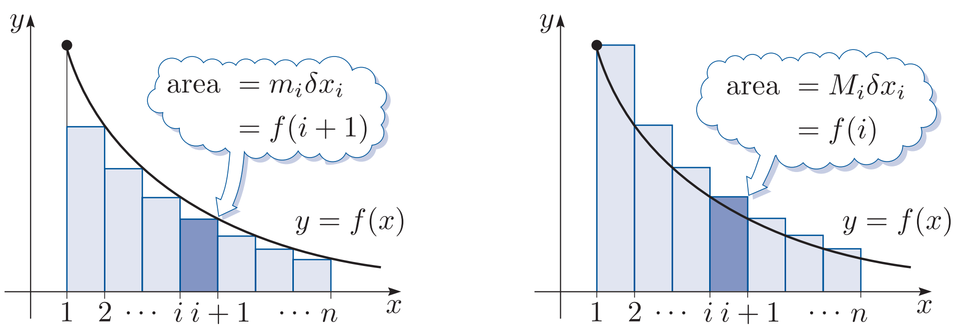


Figure 16 The lower and upper Riemann sums of f

Also, each subinterval in the partition has length 1. Hence the lower and upper Riemann sums for f on $[1, n]$ are

$$L(f, P_{n-1}) = \sum_{i=1}^{n-1} m_i \times 1 = f(2) + \cdots + f(n) = s_n - f(1)$$

and

$$U(f, P_{n-1}) = \sum_{i=1}^{n-1} M_i \times 1 = f(1) + \cdots + f(n-1) = s_n - f(n).$$

Since f is bounded and monotonic on $[1, n]$, it follows from Theorem F48 in Subsection 1.2 that the integral $I_n = \displaystyle\int_1^n f$ exists and satisfies

$$L(f, P_{n-1}) \le I_n \le U(f, P_{n-1}),$$

so

$$s_n - f(1) \le I_n \le s_n - f(n). \tag{22}$$

We now consider two cases separately.

Case (a): The sequence of integrals (I_n) is bounded above.

In this case, there exists $M \in \mathbb{R}$ such that

$$I_n \le M, \quad \text{for } n = 2, 3, \ldots.$$

It then follows from the left-hand inequality in statement (22) that

$$s_n \le f(1) + M, \quad \text{for } n = 2, 3, \ldots.$$

Thus the increasing sequence (s_n) is bounded above, so it is convergent, by the Monotone Convergence Theorem (Theorem D22 in Subsection 5.1 of Unit D2).

Hence the series $\sum_{n=1}^{\infty} f(n)$ is convergent.

Case (b): The sequence (I_n) is not bounded above.

The sequence (I_n) is increasing, since

$$I_{n+1} - I_n = \int_n^{n+1} f \ge 0,$$

so in this case

$$I_n \to \infty \text{ as } n \to \infty.$$

It follows from the right-hand inequality in statement (22) that

$$s_n \ge I_n, \quad \text{for } n = 2, 3, \ldots.$$

Thus, by the Squeeze Rule for sequences which tend to infinity (Theorem D18 in Subsection 4.3 of Unit D2),

$$s_n \to \infty \text{ as } n \to \infty.$$

Hence the series $\sum_{n=1}^{\infty} f(n)$ is divergent. ■

In Theorem D33 of Unit D3 *Series* you saw that the basic series $\sum_{n=1}^{\infty} 1/n^p$ converges for $p \ge 2$ and diverges for $0 < p \le 1$. We can now use the Integral Test to deduce the behaviour of this series for all $p > 0$. We first consider the case when $p = 1$.

Worked Exercise F38

Use the fact that $\displaystyle\int \frac{dx}{x} = \log x$ to prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Solution

Let

$$f(x) = \frac{1}{x} \quad (x \in [1, \infty)).$$

Then f is positive and decreasing on $[1, \infty)$, and

$$f(x) \to 0 \ \text{ as } x \to \infty.$$

Also, for $n \geq 1$,

$$\begin{aligned}\int_1^n f &= \int_1^n \frac{dx}{x} \\ &= \big[\log x\big]_1^n \\ &= \log n \to \infty \ \text{ as } n \to \infty.\end{aligned}$$

Hence, by part (b) of the Integral Test, the series diverges.

We now consider the series $\sum_{n=1}^{\infty} 1/n^p$ when $p > 0$ but $p \neq 1$.

Worked Exercise F39

Use the Integral Test to determine the behaviour of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \text{for } p > 0,\ p \neq 1.$$

Solution

Let $p > 0$ and $p \neq 1$, and let

$$f(x) = 1/x^p \quad (x \in [1, \infty)).$$

Then f is positive and decreasing on $[1, \infty)$, and

$$f(x) \to 0 \ \text{ as } x \to \infty.$$

Also, for $n \in \mathbb{N}$,

$$\int_1^n f = \int_1^n \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^n = \frac{n^{1-p} - 1}{1-p}. \qquad (*)$$

First suppose that $p > 1$. Then $p - 1 > 0$, so equation $(*)$ gives

$$\int_1^n f = \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right) < \frac{1}{p-1}.$$

Hence the sequence of integrals $\left(\int_1^n f\right)$ is bounded above, so it follows from part (a) of the Integral Test that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Now suppose that $0 < p < 1$. Then $1 - p > 0$, so $1/n^{1-p} \to 0$ as $n \to \infty$, and hence

$$n^{1-p} \to \infty \text{ as } n \to \infty,$$

by the Reciprocal Rule for sequences (Theorem D16 from Subsection 4.1 of Unit D2). We deduce from equation $(*)$ that

$$\int_1^n f \to \infty \text{ as } n \to \infty.$$

Hence, by part (b) of the Integral Test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

So, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$, and diverges for $0 < p < 1$.

Exercise F49

Show that

$$\int \frac{dx}{x(\log x)^2} = -\frac{1}{\log x},$$

and hence prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} \text{ is convergent.}$$

4 Stirling's Formula

This section concerns the value of the quantity $n!$ which arises in many problems in probability. You will see that integration techniques give an excellent estimate for $n!$, called Stirling's Formula, which can be expressed as

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} = 1.$$

4.1 Comparing functions of n

In this subsection we define a relation $\sim$ on the set of all positive functions with domain $\mathbb{N}$ to provide a way of comparing the behaviour of such functions for large values of n. This relation is, in fact, an equivalence relation, although we do not prove this here or make explicit use of these properties. (You met equivalence relations in Unit A3 *Mathematical language and proof* and you may like to try to prove that $\sim$ is an equivalence relation yourself; it is included as an exercise in the additional exercise booklet for this unit.)

Definition

For positive functions f and g with domain $\mathbb{N}$, we write

$$f(n) \sim g(n) \quad \text{as } n \to \infty$$

to mean

$$\frac{f(n)}{g(n)} \to 1 \quad \text{as } n \to \infty.$$

For example,

$$n^2 + n \sim n^2 \quad \text{as } n \to \infty,$$

since $n^2 + n > 0$ and $n^2 > 0$, for $n = 1, 2, \ldots$, and

$$\frac{n^2 + n}{n^2} \to 1 \quad \text{as } n \to \infty.$$

Note that the statement

$$f(n) \sim g(n) \quad \text{as } n \to \infty$$

does *not* imply that $f(n) - g(n)$ tends to zero or is even bounded. For instance, in the above example, $(n^2 + n) - n^2 = n$ tends to infinity. Note that often we omit 'as $n \to \infty$' when writing expressions of this type.

We have the following Combination Rules for $\sim$. These rules follow from the Combination Rules for sequences; we omit the proofs.

Theorem F61 Combination Rules for $\sim$

If $f_1(n) \sim g_1(n)$ and $f_2(n) \sim g_2(n)$, then:

Sum Rule	$f_1(n) + f_2(n) \sim g_1(n) + g_2(n)$
Multiple Rule	$\lambda f_1(n) \sim \lambda g_1(n)$, for $\lambda \in \mathbb{R}^+$
Product Rule	$f_1(n)f_2(n) \sim g_1(n)g_2(n)$
Quotient Rule	$\dfrac{f_1(n)}{f_2(n)} \sim \dfrac{g_1(n)}{g_2(n)}$.

Note that if $l > 0$, then the statements

$$f(n) \to l \quad \text{as } n \to \infty$$

and

$$f(n) \sim l \quad \text{as } n \to \infty$$

are equivalent, since each is equivalent to the statement

$$\frac{f(n)}{l} \to 1 \quad \text{as } n \to \infty.$$

4.2 Calculating factorials

For small values of n we can evaluate $n!$ directly by multiplication or by using a scientific calculator.

Exercise F50

Complete the following table of values of $n!$.

n	$n!$	n	$n!$	n	$n!$
1	1	6	720	20	$2.432\ldots \times 10^{18}$
2	2	7	5 040	30	
3	6	8	40 320	40	
4	24	9	362 880	50	
5	120	10	3 628 800	60	

As n increases, $n!$ grows very quickly, and is soon beyond the range of a calculator. Many calculations in probability theory involve $n!$ for large values of n, so it is important to be able to estimate this quantity as accurately as possible. The following result gives us a way of doing this.

Theorem F62 Stirling's Formula

$$n! \sim \sqrt{2\pi n}\,(n/e)^n \quad \text{as } n \to \infty.$$

We give the proof of this result in Subsection 4.3.

James Stirling (1692–1770) was a Scottish mathematician who made many contributions to analysis. He was friends with Isaac Newton and Colin Maclaurin, and at Maclaurin's request he provided help and criticisms when Maclaurin's *Treatise of Fluxions* was in proof.

Stirling's most important work was his *Methodus Differentialis*, published in 1730. It focuses largely on infinite series and was directly stimulated by earlier work of Wallis and Newton. It includes a series for $\log n!$, now usually written in the form known as Stirling's Formula, given above. Earlier in 1730, Abraham de Moivre (1667–1754), famous for his work on the laws of chance, had published a similar formula for $\log n!$, but he had not been able to find a precise value for the constant which Stirling showed to be $\sqrt{2\pi}$.

Exercise F51

Use a calculator to evaluate $\sqrt{2\pi n}\,(n/e)^n$ for the following values of n.

(a) $n = 5$ (b) $n = 10$ (c) $n = 50$

As you saw in Exercise F51, even for small values of n, Stirling's Formula gives reasonable approximations to $n!$, and the relative error (that is, the error expressed as a percentage of the true value) decreases as n increases.

n	$n!$	Stirling's approximation	Relative error
10	3 628 800	3 598 696	0.83%
20	2.433×10^{18}	2.423×10^{18}	0.42%
50	3.041×10^{64}	3.036×10^{64}	0.16%
100	9.333×10^{157}	9.325×10^{157}	0.08%

In fact it can be shown by a more careful argument that

$$e^{1/(12n+1)} \leq \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} \leq e^{1/(12n)}, \quad \text{for } n \geq 1.$$

For example, if $n = 10$, then

$$e^{1/(12n+1)} = e^{1/121} = 1.008\,29\ldots \quad \text{and} \quad e^{1/(12n)} = e^{1/120} = 1.008\,36\ldots,$$

which indicates a relative error of about 0.8%, as shown in the above table.

Stirling's Formula can be used to give estimates of probabilities. For example, if we toss n fair coins, then it can be shown that the probability of obtaining exactly r heads is $\binom{n}{r}\frac{1}{2^n}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the *binomial coefficient* which you met in Subsection 3.4 of Unit D1 (we do not prove this here). Therefore, if we toss 200 coins, then the probability of obtaining exactly 100 heads and 100 tails is

$$\begin{aligned}\binom{200}{100}\frac{1}{2^{200}} &= \frac{200!}{(100!)^2\,2^{200}}\\ &\approx \frac{\sqrt{400\pi}\,(200/e)^{200}}{\left(\sqrt{200\pi}\,(100/e)^{100}\right)^2\,2^{200}}\\ &= \frac{\sqrt{400\pi}\,\left(200^{200}/e^{200}\right)}{200\pi\,(100^{200}/e^{200})\,2^{200}}\\ &= \frac{\sqrt{400\pi}}{200\pi}\\ &= \frac{1}{10\sqrt{\pi}}\\ &= \frac{1}{17.724\ldots},\end{aligned}$$

perhaps rather higher than you might expect.

The following exercises give you a chance to practise using Stirling's Formula. In some of these exercises you will need to use the Combination Rules for $\sim$ from Subsection 4.1.

Exercise F52

Use Stirling's Formula to estimate each of the following numbers (giving your answers to two significant figures).

(a) $\binom{300}{150}\frac{1}{2^{300}}$ (b) $\frac{300!}{(100!)^3}\frac{1}{3^{300}}$

Exercise F53

Use Stirling's Formula to determine a number λ such that

$$\binom{4n}{2n} \Big/ \binom{2n}{n} \sim \lambda 2^{2n} \quad \text{as } n \to \infty.$$

Exercise F54

Use Stirling's Formula to prove that

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{1/n} = e.$$

Hint: You can assume that if $f(n) \sim g(n)$, then $(f(n))^{1/n} \sim (g(n))^{1/n}$. (This result about $\sim$ holds because if $n \in \mathbb{N}$, then

$$a \leq a^{1/n} < 1, \quad \text{for } 0 < a < 1,$$

and

$$1 < a^{1/n} \leq a, \quad \text{for } a > 1,$$

by the rules for inequalities.)

4.3 Proof of Stirling's Formula (optional)

The proof of Stirling's Formula is quite long. If you don't have time to study it properly, then you may find it interesting to skim through this subsection and see how integration is used to obtain an estimate of this type.

The idea of the proof is to consider the graph of $y = \log x$ and a sequence of small sets which lie below $y = \log x$ and above the graph of a polygonal approximation to $y = \log x$. We show that the areas of these small sets form a convergent series and deduce from this that the sequence

$$a_n = \frac{n^{n+(1/2)}}{e^{n-1}n!}, \quad n = 2, 3, \ldots,$$

is convergent. Finally we use Theorem F59(b) to find the limit of the sequence (a_n), and this gives Stirling's Formula.

Proof of Theorem F62 We need to show that

$$n! \sim \sqrt{2\pi n}\,(n/e)^n \text{ as } n \to \infty.$$

We begin the proof by considering the function

$$f(x) = \log x.$$

For $n = 2, 3, \ldots$, let P_{n-1} be the standard partition of the interval $[1, n]$ with $n - 1$ subintervals:

$$\{[1, 2], \ldots, [i, i + 1], \ldots, [n - 1, n]\}.$$

Now consider the sequence $(c_n)_2^\infty$, where c_n is the total area of the segments which lie between the graph

$$y = \log x, \quad x \in [1, n],$$

and the polygonal graph with vertices

$$(1, 0), \quad (2, \log 2), \quad (3, \log 3), \quad \ldots, \quad (n, \log n),$$

as illustrated in Figure 17. This set consists of $n-1$ thin segments.

The function f is concave (that is, its derivative is decreasing), so the line segment joining

$(i, \log i)$ to $(i+1, \log(i+1))$

lies below the graph $y = \log x$.

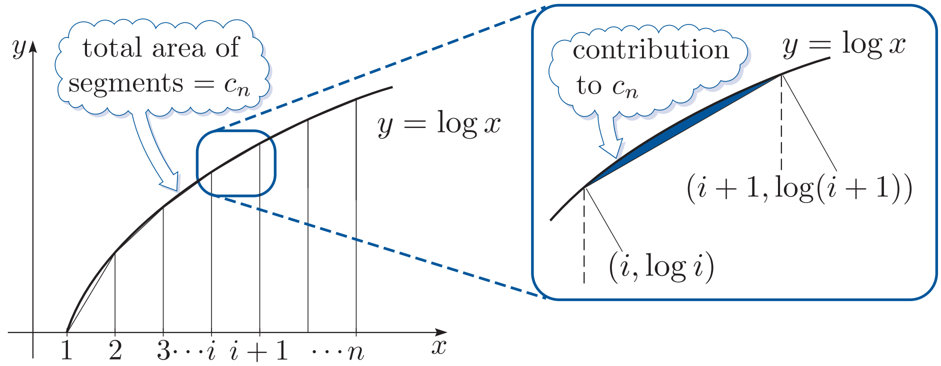


Figure 17 An approximation for the area under the graph of $\log x$

The area of the set between $y = \log x$, $x \in [1, n]$, and the x-axis is

$$\begin{aligned}\int_1^n \log x\,dx &= \left[x\log x - x\right]_1^n \\ &= n\log n - (n-1). \qquad (23)\end{aligned}$$

The area between the polygonal graph and the x-axis is

$$\tfrac{1}{2}(L(f, P_{n-1}) + U(f, P_{n-1})). \qquad (24)$$

This is because, on each subinterval $[i, i+1]$, the function f attains its minimum when $x = i$ and its maximum when $x = i+1$, so the area of the trapezium whose base is the subinterval is equal to the average of these two numbers.

Since f is increasing, we have

$$\begin{aligned}L(f, P_{n-1}) &= \log 1 + \log 2 + \cdots + \log(n-1) \\ &= \log(n-1)! \\ &= \log(n!/n) \\ &= \log n! - \log n \qquad (25)\end{aligned}$$

and

$$\begin{aligned}U(f, P_{n-1}) &= \log 2 + \log 3 + \cdots + \log n \\ &= \log n!. \qquad (26)\end{aligned}$$

Substituting from equations (25) and (26) into equation (24), we find that the area between the polygonal graph and the x-axis is

$$\tfrac{1}{2}(\log n! - \log n + \log n!) = \log n! - \tfrac{1}{2}\log n. \qquad (27)$$

It follows from equations (23) and (27) that

$$\begin{aligned} c_n &= n\log n - (n-1) - \log n! + \tfrac{1}{2}\log n \\ &= \log\left(\frac{n^{n+(1/2)}}{e^{n-1}n!}\right). \end{aligned}$$

The sequence (c_n) is positive and increasing. Also,

$$c_n \leq \log 2, \quad \text{for } n \in \mathbb{N},$$

since the $n-1$ segments which contribute to the area c_n can be translated so that they all lie (without overlapping) in the triangle with vertices $(1,0)$, $(1,\log 2)$ and $(2,\log 2)$, as illustrated in Figure 18. This geometric property holds because the function log is concave. It follows from the Monotone Convergence Theorem (Theorem D22 in Subsection 5.1 of Unit D2) that the sequence (c_n) is convergent.

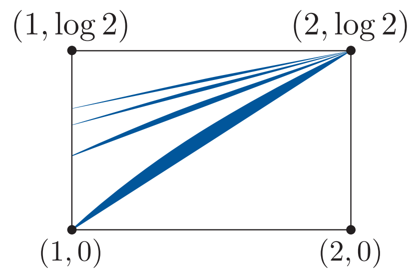


Figure 18 The segments of area c_n

Next, since the exponential function is continuous, the sequence

$$a_n = e^{c_n}, \quad n = 2, 3, \dots,$$

is convergent also. Thus

$$a_n = \frac{n^{n+(1/2)}}{e^{n-1}n!} \to L \text{ as } n \to \infty, \tag{28}$$

for some non-zero number L.

To find L, we consider the quotient

$$\frac{a_n^2}{a_{2n}} = \frac{n^{2n+1}}{e^{2n-2}(n!)^2} \Bigg/ \frac{(2n)^{2n+(1/2)}}{e^{2n-1}(2n)!} = \frac{(2n)!\, n^{1/2}}{(n!)^2\, 2^{2n}} \times \frac{e}{\sqrt{2}}.$$

We now let $n \to \infty$ in this equation. We have

$$a_n \to L, \quad a_{2n} \to L \quad \text{and} \quad \frac{(2n)!\, n^{1/2}}{(n!)^2\, 2^{2n}} \to \frac{1}{\sqrt{\pi}},$$

by Theorem F59(b).

Recall that Theorem F59(b) says that

$$\lim_{n\to\infty} \frac{(n!)^2\, 2^{2n}}{(2n)!\sqrt{n}} = \sqrt{\pi}.$$

Hence

$$\frac{L^2}{L} = \frac{1}{\sqrt{\pi}} \times \frac{e}{\sqrt{2}}, \quad \text{so} \quad L = \frac{e}{\sqrt{2\pi}}.$$

Thus we can rewrite statement (28) in the form

$$a_n = \frac{n^{n+(1/2)}}{e^{n-1}n!} \to \frac{e}{\sqrt{2\pi}} \quad \text{as } n \to \infty.$$

Hence, by the Combination Rules for sequences,

$$\frac{e^n n!}{n^n\sqrt{n}} \to \sqrt{2\pi} \quad \text{as } n \to \infty,$$

which can be rearranged to give Stirling's Formula:

$$n! \sim \sqrt{2\pi n}\,(n/e)^n \quad \text{as } n \to \infty.$$

Summary

In this unit you have studied the formal definition of what it means for a function f to be integrable on a closed interval $[a, b]$. You did this by studying lower and upper Riemann sums for f, which give lower and upper estimates for the area between the graph of f and the x-axis for $a \leq x \leq b$, by approximating the area with a sum of areas of rectangles. These rectangles are based on a collection of subintervals of $[a, b]$ known as a partition of $[a, b]$, and you saw that, by decreasing the length of these subintervals (that is, the mesh of the partition), we get increasingly accurate estimates for the area. If the supremum of the lower Riemann sums is equal to the infimum of the upper Riemann sums, then we say that the function f is integrable on $[a, b]$ and the common limit is the integral of f. You also saw that to prove that f is integrable it is sufficient to consider the sequence of standard partitions P_n where the interval is divided into n subintervals of equal lengths.

You then met the Fundamental Theorem of Calculus and saw that integration can be thought of as the process of finding a primitive, and hence as the opposite of differentiation. You studied a number of techniques for finding primitives, including integration by substitution, integration by parts and the use of a reduction formula.

You also met some useful inequalities that enable us to estimate various integrals when it is not possible to find an exact value. You saw how these can be used to prove Wallis' Formula, an approximation for π, and studied the Integral Test which gives a method based on integration for determining the convergence or divergence of certain series. Finally you met Stirling's Formula which gives an approximation for $n!$ and is proved using upper and lower Riemann sums.

Learning outcomes

After working through this unit, you should be able to:

- determine the *lower Riemann sum* $L(f, P)$ and the *upper Riemann sum* $U(f, P)$ for a given function f and *partition* P
- understand the definition of the *integral* $\int_a^b f$
- use upper and lower Riemann sums to determine whether a given function is *integrable*
- use basic rules for manipulating integrals
- state various sufficient conditions for a function to be integrable
- explain what is meant by a *primitive* of a function and understand the Fundamental Theorem of Calculus
- use the Fundamental Theorem of Calculus and the table of standard primitives to evaluate certain integrals
- use *integration by substitution* and *integration by parts*
- use the *reduction of order* method to evaluate certain integrals
- determine lower and upper estimates for given integrals
- state Wallis' Formula
- use the Integral Test to determine the convergence or divergence of certain series
- understand the connection between integration and Stirling's Formula for $n!$
- use Stirling's Formula to determine the behaviour of certain sequences involving factorials.

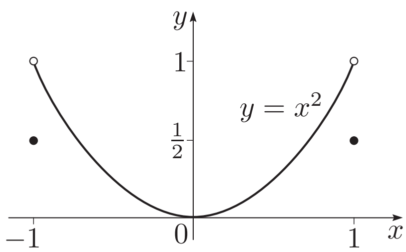
Table of standard primitives

$f(x)$	Primitive $F(x)$	Domain
$x^n,\ n \in \mathbb{Z} - \{-1\}$	$x^{n+1}/(n+1)$	$\mathbb{R}$ or $\mathbb{R} - \{0\}$
$x^\alpha,\ \alpha \neq -1$	$x^{\alpha+1}/(\alpha+1)$	$\mathbb{R}^+$
$a^x,\ a > 0$	$a^x/\log a$	$\mathbb{R}$
$\sin x$	$-\cos x$	$\mathbb{R}$
$\cos x$	$\sin x$	$\mathbb{R}$
$\tan x$	$\log(\sec x)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
e^x	e^x	$\mathbb{R}$
$1/x$	$\log x$	$(0, \infty)$
$1/x$	$\log\lvert x\rvert$	$(-\infty, 0)$
$\log x$	$x\log x - x$	$(0, \infty)$
$\sinh x$	$\cosh x$	$\mathbb{R}$
$\cosh x$	$\sinh x$	$\mathbb{R}$
$\tanh x$	$\log(\cosh x)$	$\mathbb{R}$
$(a^2 - x^2)^{-1},\ a \neq 0$	$\dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)$	$(-a, a)$
$(a^2 + x^2)^{-1},\ a \neq 0$	$\dfrac{1}{a}\tan^{-1}(x/a)$	$\mathbb{R}$
$(a^2 - x^2)^{-1/2},\ a \neq 0$	$\sin^{-1}(x/a)$	$(-a, a)$
	$-\cos^{-1}(x/a)$	$(-a, a)$
$(x^2 - a^2)^{-1/2},\ a \neq 0$	$\log(x + (x^2 - a^2)^{1/2})$	(a, ∞)
	$\cosh^{-1}(x/a)$	(a, ∞)
$(a^2 + x^2)^{-1/2},\ a \neq 0$	$\log(x + (a^2 + x^2)^{1/2})$	$\mathbb{R}$
	$\sinh^{-1}(x/a)$	$\mathbb{R}$
$(a^2 - x^2)^{1/2},\ a \neq 0$	$\frac{1}{2}x(a^2 - x^2)^{1/2} + \frac{1}{2}a^2\sin^{-1}(x/a)$	$(-a, a)$
$(x^2 - a^2)^{1/2},\ a \neq 0$	$\frac{1}{2}x(x^2 - a^2)^{1/2} - \frac{1}{2}a^2\log(x + (x^2 - a^2)^{1/2})$	(a, ∞)
$(a^2 + x^2)^{1/2},\ a \neq 0$	$\frac{1}{2}x(a^2 + x^2)^{1/2} + \frac{1}{2}a^2\log(x + (a^2 + x^2)^{1/2})$	$\mathbb{R}$
$e^{ax}\cos bx,\ a, b \neq 0$	$\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$	$\mathbb{R}$
$e^{ax}\sin bx,\ a, b \neq 0$	$\dfrac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$	$\mathbb{R}$

Solutions to exercises

Solution to Exercise F33

(a) The graph of f is shown below.



First, $\min f = 0$, since

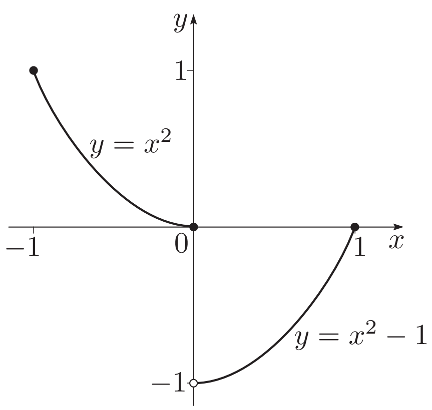
1. $f(x) \geq 0$, for all $x \in [-1, 1]$,
2. $f(0) = 0$.

Next, $\inf f = 0$, since f has minimum 0 on $[-1, 1]$.
Also, $\sup f = 1$, since

1. $f(x) \leq 1$, for all $x \in [-1, 1]$,
2. if $M' < 1$, then M' is not an upper bound for f on $[-1, 1]$ because the sequence $(1 - 1/n)$ is contained in $[-1, 1]$ and
$$f(1 - 1/n) = (1 - 1/n)^2 \to 1 \text{ as } n \to \infty,$$
so there exists $x' \in [-1, 1]$ such that $f(x') > M'$. (Alternatively, you could consider $f(-1 + 1/n)$.)

Finally, $\max f$ does not exist, since there is no point x such that $f(x) = 1$.

(b) The graph of f is shown below.



First, $\inf f = -1$, since

1. $f(x) \geq -1$, for all $x \in [-1, 1]$,
2. if $m' > -1$, then m' is not a lower bound for f on $[-1, 1]$ because the sequence $(1/n)$ is contained in $[-1, 1]$ and
$$f(1/n) = (1/n)^2 - 1 \to -1 \text{ as } n \to \infty,$$
so there exists $x' \in [-1, 1]$ such that $f(x') < m'$.

Next, $\min f$ does not exist, since there is no point x in $[-1, 1]$ such that $f(x) = -1$.
Also, $\max f = 1$, since

1. $f(x) \leq 1$, for all $x \in [-1, 1]$,
2. $f(-1) = 1$.

Finally, $\sup f = 1$, since f has maximum 1 on $[-1, 1]$.

Solution to Exercise F34

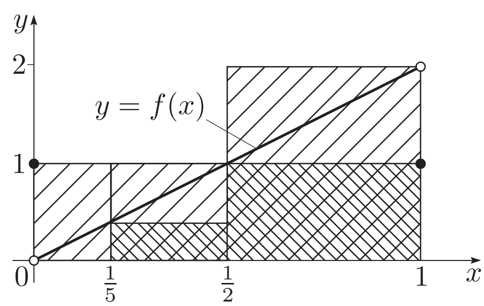
The interval $[-1, 2]$ has length 3; hence the 4 subintervals in this standard partition P of $[-1, 2]$ must each have length $\frac{3}{4}$. Thus the required standard partition of $[-1, 2]$ is

$$P = \left\{ \left[-1, -\tfrac{1}{4}\right], \left[-\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{5}{4}\right], \left[\tfrac{5}{4}, 2\right] \right\}.$$

The mesh of P is the common length of the subintervals, namely $\frac{3}{4}$.

Solution to Exercise F35

The function and the partition are illustrated below.



For the three subintervals in P, we have

$$\begin{aligned}
m_1 &= 0, & M_1 &= f(0) = 1, & \delta x_1 &= \tfrac{1}{5}, \\
m_2 &= f\left(\tfrac{1}{5}\right) = \tfrac{2}{5}, & M_2 &= f\left(\tfrac{1}{2}\right) = 1, & \delta x_2 &= \tfrac{3}{10}, \\
m_3 &= f\left(\tfrac{1}{2}\right) = 1, & M_3 &= 2, & \delta x_3 &= \tfrac{1}{2}.
\end{aligned}$$

(Note that m_3 is also equal to $f(1)$.)

Then

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{3} m_i \delta x_i \\ &= \left(0 \times \frac{1}{5}\right) + \left(\frac{2}{5} \times \frac{3}{10}\right) + \left(1 \times \frac{1}{2}\right) \\ &= 0 + \frac{3}{25} + \frac{1}{2} \\ &= \frac{31}{50} \end{aligned}$$

and

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{3} M_i \delta x_i \\ &= \left(1 \times \frac{1}{5}\right) + \left(1 \times \frac{3}{10}\right) + \left(2 \times \frac{1}{2}\right) \\ &= \frac{1}{5} + \frac{3}{10} + 1 \\ &= \frac{3}{2}. \end{aligned}$$

Solution to Exercise F36

Let

$$f(x) = x, \quad x \in [0,1],$$

and let P_n be the standard partition of $[0,1]$,

$$P_n = \left\{ \left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \ldots, \left[\frac{n-1}{n}, 1\right] \right\}.$$

Since f is increasing we have, for $i = 1, 2, \ldots, n$,

$$m_i = f\left(\frac{i-1}{n}\right) = \frac{i-1}{n},$$

$$M_i = f\left(\frac{i}{n}\right) = \frac{i}{n}$$

and

$$\delta x_i = \frac{1}{n}.$$

Thus

$$\begin{aligned} L(f,P_n) &= \sum_{i=1}^{n} m_i \delta x_i \\ &= \sum_{i=1}^{n} \left(\frac{i-1}{n} \times \frac{1}{n}\right) \\ &= \frac{1}{n^2}(1 + 2 + \cdots + (n-1)) \\ &= \frac{1}{n^2} \times \frac{(n-1)n}{2} \\ &= \frac{1}{2}\left(1 - \frac{1}{n}\right), \end{aligned}$$

and also

$$\begin{aligned} U(f,P_n) &= \sum_{i=1}^{n} M_i \delta x_i \\ &= \sum_{i=1}^{n} \left(\frac{i}{n} \times \frac{1}{n}\right) \\ &= \frac{1}{n^2}(1 + 2 + \cdots + n) \\ &= \frac{1}{n^2} \times \frac{n(n+1)}{2} \\ &= \frac{1}{2}\left(1 + \frac{1}{n}\right). \end{aligned}$$

(Here we have used the fact that the sum of the arithmetic series $1 + 2 + \cdots + n$ is $\dfrac{n(n+1)}{2}$, for each $n \in \mathbb{N}$.)

Then, as $n \to \infty$, we have

$$\lim_{n\to\infty} L(f,P_n) = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} U(f,P_n) = \frac{1}{2}.$$

In particular,

$$\frac{1}{2} \leq \underline{\int_0^1} f \leq \overline{\int_0^1} f \leq \frac{1}{2}$$

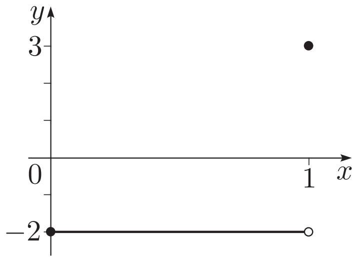
and hence

$$\underline{\int_0^1} f = \overline{\int_0^1} f = \frac{1}{2}.$$

Thus f is integrable on $[0,1]$ and $\displaystyle\int_0^1 f = \frac{1}{2}$.

Solution to Exercise F37

(a) The graph of f is shown below.



Let P_n be the standard partition of $[0,1]$ into n equal subintervals:

$$\left\{\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\ldots,\left[\frac{n-1}{n},1\right]\right\}.$$

For $i=1,2,\ldots,n$, we have

$$m_i=\inf\{f(x):(i-1)/n\le x\le i/n\}=-2.$$

For $i=1,2,\ldots,n-1$, we have

$$M_i=\sup\{f(x):(i-1)/n\le x\le i/n\}=-2.$$

Also, $M_n=3$ and

$$\delta x_i=\frac{1}{n},\quad \text{for } i=1,2,\ldots,n.$$

Hence

$$\begin{aligned}L(f,P_n)&=\sum_{i=1}^{n}m_i\,\delta x_i\\&=\sum_{i=1}^{n}\left(-2\times\frac{1}{n}\right)\\&=n\times\left(\frac{-2}{n}\right)=-2,\\U(f,P_n)&=\sum_{i=1}^{n}M_i\,\delta x_i\\&=\sum_{i=1}^{n-1}M_i\,\delta x_i+M_n\,\delta x_n\\&=\sum_{i=1}^{n-1}\left(-2\times\frac{1}{n}\right)+\left(3\times\frac{1}{n}\right)\\&=(n-1)\left(\frac{-2}{n}\right)+\frac{3}{n}\\&=-2+\frac{5}{n}\to-2\ \text{ as } n\to\infty.\end{aligned}$$

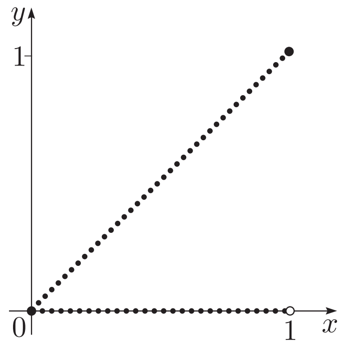
Since $\|P_n\|\to 0$ as $n\to\infty$, and

$$\lim_{n\to\infty}L(f,P_n)=\lim_{n\to\infty}U(f,P_n)=-2,$$

it follows from Theorem F45 that f is integrable on $[0,1]$ and

$$\int_0^1 f=-2.$$

(b) The graph of f is shown below.



Let P_n be the standard partition of $[0,1]$ into n equal subintervals:

$$\left\{\left[0,\frac{1}{n}\right],\left[\frac{1}{n},\frac{2}{n}\right],\ldots,\left[\frac{n-1}{n},1\right]\right\}.$$

Each subinterval

$$\left[\frac{i-1}{n},\frac{i}{n}\right],\quad \text{for } i=1,2,\ldots,n,$$

contains both rational and irrational points:

at the rational points, $f(x)=x$,
at the irrational points, $f(x)=0$.

Hence, for $i=1,2,\ldots,n$,

$$m_i=0\quad\text{and}\quad M_i=\frac{i}{n}.$$

Also,

$$\delta x_i=\frac{1}{n},\quad \text{for } i=1,2,\ldots,n.$$

Hence

$$\begin{aligned} L(f,P_n) &= \sum_{i=1}^{n} m_i\,\delta x_i \\ &= \sum_{i=1}^{n}\left(0\times\frac{1}{n}\right) = 0, \\ U(f,P_n) &= \sum_{i=1}^{n} M_i\,\delta x_i \\ &= \sum_{i=1}^{n}\left(\frac{i}{n}\times\frac{1}{n}\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} i \\ &= \frac{1}{n^2}\times\frac{n(n+1)}{2} \\ &= \frac{1}{2}+\frac{1}{2n}\to\frac{1}{2} \quad \text{as } n\to\infty. \end{aligned}$$

Since $\|P_n\|\to 0$ as $n\to\infty$, but

$$\lim_{n\to\infty} L(f,P_n) = 0 \neq \tfrac{1}{2} = \lim_{n\to\infty} U(f,P_n),$$

it follows from Theorem F46 that f is not integrable on $[0,1]$.

Solution to Exercise F38

(a) We have

$$\begin{aligned} F'(x) &= \frac{1}{x+(x^2-4)^{1/2}}\left(1+\tfrac{1}{2}(x^2-4)^{-1/2}\,2x\right) \\ &= \frac{(x^2-4)^{-1/2}\left((x^2-4)^{1/2}+x\right)}{x+(x^2-4)^{1/2}} \\ &= (x^2-4)^{-1/2} \\ &= f(x), \end{aligned}$$

as required.

(b) We have

$$\begin{aligned} F'(x) &= \frac{1}{1+\sinh^2 x}\cosh x \\ &= \frac{\cosh x}{\cosh^2 x} \\ &= \operatorname{sech} x \\ &= f(x), \end{aligned}$$

as required.

Solution to Exercise F39

(a) From the Fundamental Theorem of Calculus and the table of standard primitives, we deduce that

$$\begin{aligned} &\int_0^4 (x^2+9)^{1/2}\,dx \\ &= \left[\tfrac{1}{2}x(x^2+9)^{1/2}+\tfrac{9}{2}\log(x+(x^2+9)^{1/2})\right]_0^4 \\ &= 10+\tfrac{9}{2}\log 9-\tfrac{9}{2}\log 3 \\ &= 10+\tfrac{9}{2}\log 3. \end{aligned}$$

(b) From the Fundamental Theorem of Calculus and the table of standard primitives, we deduce that

$$\begin{aligned} \int_1^e \log x\,dx &= \left[x\log x - x\right]_1^e \\ &= (e-e)-(0-1) = 1. \end{aligned}$$

Solution to Exercise F40

Using the table of standard primitives and the Combination Rules, we obtain the following primitives.

(a) $$\begin{aligned} F(x) &= 4(x\log x - x) - 2\left(\tfrac{1}{2}\tan^{-1}(x/2)\right) \\ &= 4(x\log x - x) - \tan^{-1}(x/2) \end{aligned}$$

(b) $$\begin{aligned} F(x) &= 2\left(\tfrac{1}{3}\log(\sec 3x)\right) + \frac{e^{2x}}{2^2+1^2}(2\cos x+\sin x) \\ &= \tfrac{2}{3}\log(\sec 3x) + \tfrac{1}{5}e^{2x}(2\cos x+\sin x) \end{aligned}$$

(These results can be checked by differentiation.)

Solution to Exercise F41

(a) Take $u=\sin 3x$; then

$$\frac{du}{dx} = 3\cos 3x, \quad \text{so} \quad du = 3\cos 3x\,dx.$$

Hence

$$\begin{aligned} \int \sin(\sin 3x)\cos 3x\,dx &= \tfrac{1}{3}\int \sin u\,du \\ &= -\tfrac{1}{3}\cos u \\ &= -\tfrac{1}{3}\cos(\sin 3x). \end{aligned}$$

(b) Taking $u=2+3x^3$, we obtain

$$\int x^2(2+3x^3)^7\,dx = \tfrac{1}{72}(2+3x^3)^8.$$

(c) Taking $u = 2x^2$, we obtain

$$\int x\sin(2x^2)\,dx = -\tfrac{1}{4}\cos(2x^2).$$

(d) Using equation (17), we obtain

$$\int x/(2+3x^2)\,dx = \tfrac{1}{6}\log(2+3x^2).$$

Solution to Exercise F42

Let $u = 1 + e^x$; then

$$\frac{du}{dx} = e^x, \quad \text{so} \quad du = e^x\,dx.$$

Also,

$$\begin{aligned} &\text{when } x = 0, \quad u = 2,\\ &\text{when } x = 1, \quad u = 1 + e. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^1 \frac{e^x}{(1+e^x)^2}\,dx &= \int_2^{1+e} \frac{du}{u^2}\\ &= \left[\frac{-1}{u}\right]_2^{1+e}\\ &= -\frac{1}{1+e} + \frac{1}{2} = \frac{e-1}{2(1+e)}. \end{aligned}$$

Solution to Exercise F43

(a) Let $u = (x-1)^{1/2}$, so $x = u^2 + 1$; then

$$\frac{dx}{du} = 2u, \quad \text{so} \quad dx = 2u\,du.$$

Hence

$$\begin{aligned} &\int \frac{dx}{3(x-1)^{3/2} + x(x-1)^{1/2}}\\ &= \int \frac{2u}{3u^3 + (u^2+1)u}\,du\\ &= \int \frac{2}{4u^2+1}\,du\\ &= 2\int \frac{du}{(2u)^2+1}\\ &= \tan^{-1}(2u)\\ &= \tan^{-1}(2(x-1)^{1/2}). \end{aligned}$$

(b) Let $u = \sqrt{1+e^x}$, so $x = \log(u^2-1)$; then

$$\frac{dx}{du} = \frac{2u}{u^2-1}, \quad \text{so} \quad dx = \frac{2u}{u^2-1}\,du.$$

Also,

$$\begin{aligned} &\text{when } x = 0, \quad u = \sqrt{2},\\ &\text{when } x = \log 3, \quad u = 2. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{\log 3} e^x\sqrt{1+e^x}\,dx &= \int_{\sqrt{2}}^2 (u^2-1)u\frac{2u}{u^2-1}\,du\\ &= \int_{\sqrt{2}}^2 2u^2\,du\\ &= \left[\tfrac{2}{3}u^3\right]_{\sqrt{2}}^2\\ &= (16 - 4\sqrt{2})/3. \end{aligned}$$

Solution to Exercise F44

(a) Here we use integration by parts, with

$$f(x) = \log x \quad \text{and} \quad g'(x) = x^{1/3};$$

then

$$f'(x) = 1/x \quad \text{and} \quad g(x) = \tfrac{3}{4}x^{4/3}.$$

Hence

$$\begin{aligned} \int x^{1/3}\log x\,dx &= \tfrac{3}{4}x^{4/3}\log x - \tfrac{3}{4}\int x^{4/3}x^{-1}\,dx\\ &= \tfrac{3}{4}x^{4/3}\log x - \tfrac{3}{4}\int x^{1/3}\,dx\\ &= \tfrac{3}{4}x^{4/3}\log x - \tfrac{9}{16}x^{4/3}. \end{aligned}$$

(b) We use integration by parts twice. On each occasion we differentiate the power function and integrate the trigonometric function.

We have

$$\begin{aligned} \int_0^{\pi/2} x^2\cos x\,dx &= \left[x^2\sin x\right]_0^{\pi/2}\\ &\quad - \int_0^{\pi/2} 2x\sin x\,dx\\ &= \frac{\pi^2}{4} - 2\int_0^{\pi/2} x\sin x\,dx \end{aligned}$$

and

$$\begin{aligned} \int_0^{\pi/2} x\sin x\,dx &= \left[x(-\cos x)\right]_0^{\pi/2}\\ &\quad - \int_0^{\pi/2} (-\cos x)\,dx\\ &= 0 + \left[\sin x\right]_0^{\pi/2} = 1. \end{aligned}$$

It follows that

$$\int_0^{\pi/2} x^2 \cos x \, dx = \frac{\pi^2}{4} - 2.$$

Solution to Exercise F45

(a) $I_0 = \displaystyle\int_0^1 e^x \, dx = [e^x]_0^1 = e - 1.$

(b) Using integration by parts, we obtain

$$\begin{aligned} I_n &= \int_0^1 e^x x^n \, dx \\ &= \Big[e^x x^n\Big]_0^1 - \int_0^1 e^x n x^{n-1} \, dx \\ &= e - nI_{n-1}, \quad \text{for } n \geq 1. \end{aligned}$$

(c) Using the solution to part (b) with $n = 1, 2, 3, 4$ in turn, we obtain

$$\begin{aligned} I_1 &= e - I_0 = e - (e-1) = 1, \\ I_2 &= e - 2I_1 = e - 2, \\ I_3 &= e - 3I_2 = e - 3(e-2) = 6 - 2e, \\ I_4 &= e - 4I_3 = e - 4(6-2e) = 9e - 24. \end{aligned}$$

Solution to Exercise F46

(a) Since $\sin(1/x^{10}) \leq 1$, we have

$$x \sin(1/x^{10}) \leq x, \quad \text{for } x \in [1, 3].$$

Thus it follows from Inequality Rule (a) that

$$\begin{aligned} \int_1^3 x \sin(1/x^{10}) \, dx &\leq \int_1^3 x \, dx \\ &= \left[\tfrac{1}{2}x^2\right]_1^3 \\ &= \tfrac{1}{2}(9-1) \\ &= 4. \end{aligned}$$

(b) If $x \in \left[0, \frac{1}{2}\right]$, then

$$1 = e^0 \leq e^{x^2} \leq e^{(1/2)^2} = e^{1/4},$$

because the function $x \longmapsto e^{x^2}$ is increasing on $\left[0, \frac{1}{2}\right]$.

Thus it follows from Inequality Rule (b) that

$$\tfrac{1}{2} \leq \int_0^{1/2} e^{x^2} \, dx \leq \tfrac{1}{2} e^{1/4}.$$

Solution to Exercise F47

(a) Since

$$|\sin(1/x)| \leq 1, \quad \text{for } x \in [1, 4],$$

and

$$2 + \cos(1/x) \geq 1, \quad \text{for } x \in [1, 4],$$

it follows that

$$\left|\frac{\sin(1/x)}{2 + \cos(1/x)}\right| \leq 1, \quad \text{for } x \in [1, 4].$$

Hence, by the Triangle Inequality for integrals,

$$\left|\int_1^4 \frac{\sin(1/x)}{2 + \cos(1/x)} \, dx\right| \leq \int_1^4 dx = 4 - 1 = 3.$$

(b) Since

$$\tan x \geq 0, \quad \text{for } x \in [0, \pi/4],$$

and

$$3 - \sin(x^2) \geq 2, \quad \text{for } x \in [0, \pi/4],$$

it follows that

$$0 \leq \frac{\tan x}{3 - \sin(x^2)} \leq \tfrac{1}{2} \tan x, \quad \text{for } x \in [0, \pi/4].$$

Hence, by the Triangle Inequality for integrals and Inequality Rule (a),

$$\begin{aligned} \left|\int_0^{\pi/4} \frac{\tan x}{3 - \sin(x^2)} \, dx\right| &\leq \int_0^{\pi/4} \left|\frac{\tan x}{3 - \sin(x^2)}\right| dx \\ &\leq \int_0^{\pi/4} \tfrac{1}{2} \tan x \, dx \\ &= \left[\tfrac{1}{2} \log(\sec x)\right]_0^{\pi/4} \\ &= \tfrac{1}{2}(\log(\sec \pi/4) - \log 1) \\ &= \tfrac{1}{2} \log(\sqrt{2}) \\ &= \tfrac{1}{4} \log 2. \end{aligned}$$

Solution to Exercise F48

(a) We have

$$\begin{aligned}
a_1 &= \frac{2}{1}\cdot\frac{2}{3} = \frac{4}{3},\\
a_2 &= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5} = \frac{64}{45},\\
a_3 &= \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7} = \frac{256}{175},\\
b_1 &= \frac{(1!)^2\,2^2}{2!\sqrt{1}} = 2,\\
b_2 &= \frac{(2!)^2\,2^4}{4!\sqrt{2}} = \frac{4}{3}\sqrt{2},\\
b_3 &= \frac{(3!)^2\,2^6}{6!\sqrt{3}} = \frac{16}{15}\sqrt{3}.
\end{aligned}$$

(b) Using the results of part (a),

$$\begin{aligned}
b_1^2 &= 4 = 3a_1,\\
b_2^2 &= \frac{32}{9} = \frac{5}{2}a_2,\\
b_3^2 &= \frac{256}{75} = \frac{7}{3}a_3,
\end{aligned}$$

as required.

(c) We have

$$b_n^2 = \frac{(n!)^4\,2^{4n}}{((2n)!)^2\,n}.$$

We now try to express

$$a_n = \frac{2\cdot 2\cdot 4\cdot 4\cdot\,\cdots\,\cdot(2n)(2n)}{1\cdot 3\cdot 3\cdot 5\cdot\,\cdots\,\cdot(2n-1)(2n+1)}$$

in terms of factorials.

The numerator is a product of $2n$ even numbers. Taking a factor 2 from each term, we deduce that

$$\begin{aligned}
&2\cdot 2\cdot 4\cdot 4\cdot\,\cdots\,\cdot(2n)(2n)\\
&\quad= 2^{2n}(1\cdot 1\cdot 2\cdot 2\cdot\,\cdots\,\cdot n\cdot n) = 2^{2n}(n!)^2.
\end{aligned}$$

The denominator of a_n cannot be treated in quite the same way, as all its factors are odd. To relate it to factorials, we introduce the missing even factors:

$$\begin{aligned}
&1\cdot 3\cdot 3\cdot 5\cdot 5\cdot\,\cdots\,\cdot(2n-1)(2n+1)\\
&= \frac{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdot\,\cdots\,\cdot(2n-1)(2n)(2n)(2n+1)}{2\cdot 2\cdot 4\cdot 4\cdot\,\cdots\,\cdot(2n)(2n)}\\
&= \frac{((2n)!)^2\,(2n+1)}{2^{2n}(n!)^2}.
\end{aligned}$$

It follows that

$$\begin{aligned}
a_n &= 2^{2n}(n!)^2 \Big/ \frac{((2n)!)^2\,(2n+1)}{2^{2n}(n!)^2}\\
&= \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)}\\
&= b_n^2\left(\frac{n}{2n+1}\right).
\end{aligned}$$

Hence

$$b_n^2 = \left(\frac{2n+1}{n}\right)a_n,$$

as required.

Solution to Exercise F49

Let $u = \log x$; then

$$\frac{du}{dx} = \frac{1}{x}, \quad \text{so} \quad du = \frac{dx}{x}.$$

Hence

$$\int \frac{dx}{x(\log x)^2} = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\log x}.$$

Now let

$$f(x) = \frac{1}{x(\log x)^2} \quad (x \in [2,\infty)).$$

Then f is positive and decreasing on $[2,\infty)$, and

$$f(x) \to 0 \text{ as } x \to \infty.$$

Also, for $n \geq 2$,

$$\begin{aligned}
\int_2^n f &= \int_2^n \frac{dx}{x(\log x)^2}\\
&= \left[-\frac{1}{\log x}\right]_2^n\\
&= \frac{1}{\log 2} - \frac{1}{\log n} \leq \frac{1}{\log 2}.
\end{aligned}$$

Since the sequence of integrals $\left(\int_2^n f\right)_2^\infty$ is bounded above, it follows from part (a) of the Integral Test that the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n(\log n)^2}$ converges.

Solution to Exercise F50

The values are as follows.

n	$n!$
30	$2.652\ldots \times 10^{32}$
40	$8.159\ldots \times 10^{47}$
50	$3.041\ldots \times 10^{64}$
60	$8.320\ldots \times 10^{81}$

Solution to Exercise F51

The values are as follows.

n	$\sqrt{2\pi n}\,(n/e)^n$
5	$118.019\ldots$
10	$3.598\ldots \times 10^{6}$
50	$3.036\ldots \times 10^{64}$

Solution to Exercise F52

In each part we approximate the factorials using Stirling's Formula.

(a)
$$\begin{aligned}\binom{300}{150}\frac{1}{2^{300}} &= \frac{300!}{150!\,150!\,2^{300}}\\ &\approx \frac{\sqrt{600\pi}\,(300/e)^{300}}{\left(\sqrt{300\pi}\,(150/e)^{150}\right)^2 2^{300}}\\ &= \frac{\sqrt{600\pi}}{300\pi}\\ &= \frac{\sqrt{6}}{30\sqrt{\pi}}\\ &= 0.046 \quad \text{(to 2 s.f.)}.\end{aligned}$$

(b)
$$\begin{aligned}\frac{300!}{(100!)^3}\frac{1}{3^{300}} &\approx \frac{\sqrt{600\pi}\,(300/e)^{300}}{(200\pi)^{3/2}\,(100/e)^{300}\,3^{300}}\\ &= \frac{\sqrt{600\pi}}{(200\pi)^{3/2}}\\ &= \frac{\sqrt{3}}{200\pi}\\ &= 0.0028 \quad \text{(to 2 s.f.)}.\end{aligned}$$

Solution to Exercise F53

We have
$$\binom{4n}{2n} = \frac{(4n)!}{((2n)!)^2} \quad \text{and} \quad \binom{2n}{n} = \frac{(2n)!}{(n!)^2},$$
so
$$\binom{4n}{2n}\bigg/\binom{2n}{n} = \frac{(4n)!(n!)^2}{((2n)!)^3}.$$
By Stirling's Formula, and the Product and Quotient Rules for $\sim$, we obtain
$$\begin{aligned}\binom{4n}{2n}\bigg/\binom{2n}{n} &\sim \frac{\sqrt{8\pi n}\,(4n/e)^{4n}\left(\sqrt{2\pi n}\,(n/e)^n\right)^2}{\left(\sqrt{4\pi n}\,(2n/e)^{2n}\right)^3}\\ &= \frac{\sqrt{8\pi n}\,4^{4n}\,2\pi n}{(4\pi n)^{3/2}\,2^{6n}}\\ &= \frac{2\sqrt{8}}{8}\frac{4^{4n}}{2^{6n}}\\ &= \frac{1}{\sqrt{2}}\,2^{2n}.\end{aligned}$$
Hence $\lambda = 1/\sqrt{2}$.

Solution to Exercise F54

Using Stirling's Formula, we obtain
$$\frac{n^n}{n!} \sim \frac{n^n}{\sqrt{2\pi n}\,(n/e)^n} = \frac{e^n}{\sqrt{2\pi n}}.$$
Thus, by the hint,
$$\left(\frac{n^n}{n!}\right)^{1/n} \sim \left(\frac{e^n}{\sqrt{2\pi n}}\right)^{1/n} = \frac{e}{\left(\sqrt{2\pi}\right)^{1/n}\sqrt{n^{1/n}}}.$$
We know (from Worked Exercise D26 and Exercise D34 in Subsection 3.3 of Unit D2) that, for any positive number a,
$$a^{1/n} \to 1 \text{ as } n \to \infty$$
and
$$n^{1/n} \to 1 \text{ as } n \to \infty.$$
Hence
$$\left(\sqrt{2\pi}\right)^{1/n} \to 1 \text{ as } n \to \infty$$
and
$$\sqrt{n^{1/n}} \to 1 \text{ as } n \to \infty.$$

It follows that

$$\left(\frac{n^n}{n!}\right)^{1/n} \sim e \quad \text{as } n \to \infty;$$

that is,

$$\left(\frac{n^n}{n!}\right)^{1/n} \to e \quad \text{as } n \to \infty.$$